High-ordered Random Walks and Generalized Laplacians on Hypergraphs

Linyuan Lu and Xing Peng

University of South Carolina
WAW 2011, Atlanta, GA

May 28 2011
1. Background
1 Background

2 $s$-walks on hypergraphs
Outline

1 Background

2 $s$-walks on hypergraphs

3 The definition of $s$-th Laplacian
1. Background

2. $s$-walks on hypergraphs

3. The definition of $s$-th Laplacian

4. Applications
Outline

1. Background
2. $s$-walks on hypergraphs
3. The definition of $s$-th Laplacian
4. Applications
5. Open problems
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.

$A$ is the adjacency matrix. $d_v$ is the degree of $v$. $T$ is a diagonal matrix such that $d(v, v) = d_v$. The Laplacian $L$ of $G$ is $\mathbf{I} - \frac{1}{2}AT^{-1} - \frac{1}{2}AT^{-1}$. $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $L$. We know $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n-1$ and $\lambda_1 > 0$ iff $G$ is connected.
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
- $d_v$ is the degree of $v$. 
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
- $d_v$ is the degree of $v$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$. 

\[
L = I - T - \frac{1}{2}AT - \frac{1}{2}A^T
\]

$\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $L$.
We know $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n-1$ and $\lambda_1 > 0$ iff $G$ is connected.
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
- $d_v$ is the degree of $v$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$.
- The Laplacian $\mathcal{L}$ of $G$ is $I - T^{-1/2}AT^{-1/2}$.
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
- $d_v$ is the degree of $v$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$.
- The Laplacian $\mathcal{L}$ of $G$ is $I - T^{-1/2}AT^{-1/2}$.
- $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}$.
Laplacians of graphs

- $G$ is a (undirected) graph on $n$ vertices.
- $A$ is the adjacency matrix.
- $d_v$ is the degree of $v$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$.
- The Laplacian $\mathcal{L}$ of $G$ is $I - T^{-1/2}AT^{-1/2}$.
- $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}$.
- We know $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n - 1$ and $\lambda_1 > 0$ iff $G$ is connected.
Laplacians of graphs

\( \lambda_1 \) is related to many graph parameters.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.

- The mixing rate of random walks.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.

- The mixing rate of random walks.
- The graph diameter.
Laplacians of graphs

\( \lambda_1 \) is related to many graph parameters.

- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.
- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
- The Cheeger constant.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.

- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
- The Cheeger constant.
- The isoperimetric inequalities.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.

- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
- The Cheeger constant.
- The isoperimetric inequalities.
- The expander graphs.
Laplacians of graphs

$\lambda_1$ is related to many graph parameters.

- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
- The Cheeger constant.
- The isoperimetric inequalities.
- The expander graphs.
- Quasi-random graphs.
Laplacians of graphs

\( \lambda_1 \) is related to many graph parameters.
- The mixing rate of random walks.
- The graph diameter.
- The neighborhood expansion.
- The Cheeger constant.
- The isoperimetric inequalities.
- The expander graphs.
- Quasi-random graphs.
- More applications.
A question

How do we define Laplacians for $r$-uniform hypergraphs?
Some known definitions of Laplacians

- 1993, Chung gave a first definition for regular hypergraphs.
Some known definitions of Laplacians

- 1993, Chung gave a first definition for regular hypergraphs.
- 2009, Rodríguez gave another definition.
Notations

- $H$ is an $r$-uniform hypergraph.
Notations

- $H$ is an $r$-uniform hypergraph.
- $V(H)$ and $E(H)$ are the vertex set and edge set of $H$. 

$S \subset V(H)$ such that $|S| < r$, the neighborhood $\Gamma(S)$ is 
\[ \begin{cases} 
T & \text{if } S \cap T = \emptyset \text{ and } S \cup T \text{ is an edge in } H \\
|T| = r - |S| 
\end{cases} \]

The degree of $S$, denoted by $d_S$, is $|\Gamma(S)|$.

$1 \leq s \leq r/2$, loose case.
$r/2 < s \leq r - 1$, tight case.
Notations

- $H$ is an $r$-uniform hypergraph.
- $V(H)$ and $E(H)$ are the vertex set and edge set of $H$.
- $S \subset V(H)$ such that $|S| < r$, the neighborhood $\Gamma(S)$ is \{\(T : S \cap T = \emptyset \text{ and } S \cup T \text{ is an edge in } H\}\}, here $|T| = r - |S|$.
Notations

- $H$ is an $r$-uniform hypergraph.
- $V(H)$ and $E(H)$ are the vertex set and edge set of $H$.
- $S \subset V(H)$ such that $|S| < r$, the neighborhood $\Gamma(S)$ is $\{T | S \cap T = \emptyset$ and $S \cup T$ is an edge in $H\}$, here $|T| = r - |S|$.
- The degree of $S$, denoted by $d_S$, is $|\Gamma(S)|$. 
Notations

- $H$ is an $r$-uniform hypergraph.
- $V(H)$ and $E(H)$ are the vertex set and edge set of $H$.
- $S \subset V(H)$ such that $|S| < r$, the neighborhood $\Gamma(S)$ is $\{T | S \cap T = \emptyset \text{ and } S \cup T \text{ is an edge in } H\}$, here $|T| = r - |S|$.
- The degree of $S$, denoted by $d_S$, is $|\Gamma(S)|$.
- $1 \leq s \leq r/2$, loose case.
Notations

- $H$ is an $r$-uniform hypergraph.
- $V(H)$ and $E(H)$ are the vertex set and edge set of $H$.
- $S \subset V(H)$ such that $|S| < r$, the neighborhood $\Gamma(S)$ is
  $\{T | S \cap T = \emptyset \text{ and } S \cup T \text{ is an edge in } H\}$, here $|T| = r - |S|$.
- The degree of $S$, denoted by $d_S$, is $|\Gamma(S)|$.
- $1 \leq s \leq r/2$, loose case.
- $r/2 < s \leq r - 1$, tight case.
For $1 \leq s \leq r - 1$, an $s$-walk of length $k$ is a sequence of vertices

$$v_1, v_2, \ldots, v_j, \ldots, v_{(r-s)(k-1)+r}$$

together with a sequence of edges $F_1, F_2, \ldots, F_k$ such that

$$F_i = \{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\}$$

for $1 \leq i \leq k$. 

s-walks on hypergraphs
Examples of $s$-th walks

- 1-walk of length 3 on a 3-uniform hypergraph. $r = 3$ and $s = 1$. 
Examples of $s$-th walks

- 1-walk of length 3 on a 3-uniform hypergraph. $r = 3$ and $s = 1$. 

\[
\begin{array}{ccc}
V_1 & V_2 & V_3 \\
\end{array}
\]
Examples of $s$-th walks

- 1-walk of length 3 on a 3-uniform hypergraph. $r = 3$ and $s = 1$. 

![Diagram of a 3-uniform hypergraph with vertices V1, V2, V3, V4, V5]
Examples of $s$-th walks

- 1-walk of length 3 on a 3-uniform hypergraph. $r = 3$ and $s = 1$. 
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$. 
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$.

$\begin{array}{ccc}
V_1 & V_2 & V_3 \\
\end{array}$
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$. 

![Diagram showing a 2-walk of length 4 on a 3-uniform hypergraph with vertices $V_1, V_2, V_3, V_4$.](image-url)
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$. 

![Diagram showing a 2-walk of length 4 on a 3-uniform hypergraph]
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$. 
Examples of $s$-th walks

- 2-walk of length 3 on a 4-uniform hypergraph. $r = 4$ and $s = 2$. 
Examples of $s$-th walks

- 2-walk of length 3 on a 4-uniform hypergraph. $r = 4$ and $s = 2$.
Examples of $s$-th walks

- 2-walk of length 3 on a 4-uniform hypergraph. $r = 4$ and $s = 2.$
Examples of $s$-th walks

- 2-walk of length 3 on a 4-uniform hypergraph. $r = 4$ and $s = 2$. 
$s$-path, $s$-cycle

Similar concepts:
$s$-path, $s$-cycle

Similar concepts:

- $s$-path, $s$-cycle.
s-path, s-cycle

Similar concepts:
- s-path, s-cycle.
- Hamilton s-cycle.
For each $i$ in $\{0, 1, \ldots, k\}$, the $i$-th stop $x_i$ of the $s$-walk is the ordered $s$-tuple
\[
(v_{r-s}i+1, v_{r-s}i+2, \ldots, v_{r-s}i+s).
\]
For each $i$ in $\{0, 1, \ldots, k\}$, the $i$-th stop $x_i$ of the $s$-walk is the ordered $s$-tuple 
$$(v_{(r-s)i+1}, v_{(r-s)i+2}, \ldots, v_{(r-s)i+s}).$$

The initial stop is $x_0$, and the terminal stop is $x_k$. 

Examples of $s$-th walks

- 1-walk of length 3 on a 3-uniform hypergraph. $r = 3$ and $s = 1$.
- $v_1$ is the initial stop $x_0$.
- $v_3$ is the 1-st stop $x_1$.
- $v_5$ is the 2-nd stop $x_2$.
- $v_7$ is the terminal stop $x_3$. 
Examples of $s$-th walks

- 2-walk of length 4 on a 3-uniform hypergraph. $r = 3$ and $s = 2$.
- $(v_1, v_2)$ is the initial stop $x_0$.
- $(v_2, v_3)$ is the 1-st stop $x_1$.
- $(v_3, v_4)$ is the 2-nd stop $x_2$.
- $(v_4, v_5)$ is the 3-rd stop $x_3$.
- $(v_5, v_6)$ is the terminal stop $x_4$. 
Examples of $s$-th walks

- 2-walk of length 3 on a 4-uniform hypergraph. $r = 4$ and $s = 2$.
- $(v_1, v_2)$ is the initial stop $x_0$.
- $(v_3, v_4)$ is the 1-st stop $x_1$.
- $(v_5, v_6)$ is the 2-nd stop $x_2$.
- $(v_7, v_8)$ is the terminal stop $x_3$. 
Random $s$-walks on hypergraphs

- At each step $i$, $S_i$ is the $i$-th stop.
Random $s$-walks on hypergraphs

- At each step $i$, $S_i$ is the $i$-th stop.
- A random $(r - s)$-set $T$ from $\Gamma(S)$ uniformly and add its vertices to the sequence.
Random $s$-walks on hypergraphs

- At each step $i$, $S_i$ is the $i$-th stop.
- A random $(r - s)$-set $T$ from $\Gamma(S)$ uniformly and add its vertices to the sequence.
For $0 \leq \alpha \leq 1$, an $\alpha$-lazy random $s$-walk is
\(\alpha\)-lazy random \(s\)-walk

For \(0 \leq \alpha \leq 1\), an \(\alpha\)-lazy random \(s\)-walk is

- with probability \(\alpha\), stay
\( \alpha \)-lazy random \( s \)-walk

For \( 0 \leq \alpha \leq 1 \), an \( \alpha \)-lazy random \( s \)-walk is

- with probability \( \alpha \), stay
- with probability \( 1 - \alpha \), walk
For $0 \leq \alpha \leq 1$, an $\alpha$-lazy random $s$-walk is
- with probability $\alpha$, stay
- with probability $1 - \alpha$, walk

The transition matrix is

$$\alpha I + (1 - \alpha)P.$$
s-th Laplacian for the case $1 \leq s \leq r/2$

- For $1 \leq s \leq r/2$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$. 
For $1 \leq s \leq r/2$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

$G^{(s)}$ is a weighted graph over $V^s$. 

\[ \text{The } s\text{-th Laplacian } L^{(s)} \text{ is defined to be the Laplacian of } G^{(s)}. \]
$s$-th Laplacian for the case $1 \leq s \leq r/2$

- For $1 \leq s \leq r/2$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.
- $G^{(s)}$ is a weighted graph over $V^s$.
- The weight $w(x, y) = |\{F \in E(H) : [x] \cup [y] \subseteq F\}|$. Here, $[x]$ is the set of all coordinates of $x$. 
For $1 \leq s \leq r/2$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

$G^{(s)}$ is a weighted graph over $V^s$.

The weight $w(x, y) = |\{ F \in E(H) : [x] \sqcup [y] \subseteq F \}|$. Here, $[x]$ is the set of all coordinates of $x$.

If $[x] \cap [y] \neq \emptyset$, then $w(x, y) = 0$. 

$s$-th Laplacian for the case $1 \leq s \leq r/2$
For $1 \leq s \leq r/2$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

$G^{(s)}$ is a weighted graph over $V^s$.

The weight $w(x, y) = |\{ F \in E(H) : [x] \sqcup [y] \subseteq F \}|$. Here, $[x]$ is the set of all coordinates of $x$.

If $[x] \cap [y] \neq \emptyset$, then $w(x, y) = 0$.

The $s$-th Laplacian $\mathcal{L}^{(s)}$ is defined to be the Laplacian of $G^{(s)}$. 
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
- $w : V \times V \rightarrow \mathbb{R}^{\geq 0}$ is the weight function.
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
- $w : V \times V \rightarrow \mathbb{R}^{\geq 0}$ is the weight function.
- $A$ is the adjacency matrix, i.e., $A(u, v) = w(u, v)$. 
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
- $w: V \times V \rightarrow \mathbb{R}^{\geq 0}$ is the weight function.
- $A$ is the adjacency matrix, i.e., $A(u, v) = w(u, v)$.
- The degree $d_v$ of $v$ is $\sum_u w(u, v)$. 
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
- $w: V \times V \to \mathbb{R}_{\geq 0}$ is the weight function.
- $A$ is the adjacency matrix, i.e., $A(u, v) = w(u, v)$.
- The degree $d_v$ of $v$ is $\sum_u w(u, v)$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$. 
Laplacians of weighted graphs

- $G$ is a weighted (undirected) graph on $n$ vertices.
- $w : V \times V \rightarrow \mathbb{R}^{\geq 0}$ is the weight function.
- $A$ is the adjacency matrix, i.e., $A(u, v) = w(u, v)$.
- The degree $d_v$ of $v$ is $\sum_u w(u, v)$.
- $T$ is a diagonal matrix such that $d(v, v) = d_v$.
- The Laplacian $\mathcal{L}$ of $G$ is $I - T^{-1/2}AT^{-1/2}$.
For $r/2 < s \leq r - 1$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$. 

$s$-th Laplacian for the case $r/2 < s \leq r - 1$
$s$-th Laplacian for the case $r/2 < s \leq r - 1$

For $r/2 < s \leq r - 1$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

$D^{(s)}$ is a directed graph over $V^s$. 
For $r/2 < s \leq r - 1$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

$D^{(s)}$ is a directed graph over $V^s$.

$x, y \in V^s, x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$,

$(x, y) \in E(D^{(s)})$ if $x_{r-s+j} = y_j$ for $1 \leq j \leq 2s - r$ and

$[x] \cup [y]$ is an edge of $H$. 

$s$-th Laplacian for the case $r/2 < s \leq r - 1$
For \( r/2 < s \leq r - 1 \), let \( V^s \) be the set of all (ordered) \( s \)-tuples consisting of \( s \) distinct elements in \( V \).

\( D^{(s)} \) is a directed graph over \( V^s \).

\( x, y \in V^s, \ x = (x_1, \ldots, x_s) \) and \( y = (y_1, \ldots, y_s) \),

\((x, y) \in E(D^{(s)}) \) if \( x_{r-s+j} = y_j \) for \( 1 \leq j \leq 2s - r \) and

\([x] \cup [y] \) is an edge of \( H \).

\( D^{(s)} \) is Eulerian, i.e., \( d^+_x = d^-_x \) for each \( x \in V^s \).
For $r/2 < s \leq r - 1$, let $V^s$ be the set of all (ordered) $s$-tuples consisting of $s$ distinct elements in $V$.

- $D^{(s)}$ is a directed graph over $V^s$.
  
  - $x, y \in V^s$, $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, $(x, y) \in E(D^{(s)})$ if $x_{r-s+j} = y_j$ for $1 \leq j \leq 2s - r$ and $[x] \cup [y]$ is an edge of $H$.

- $D^{(s)}$ is Eulerian, i.e., $d_x^+ = d_x^-$ for each $x \in V^s$.

The $s$-th Laplacian $\mathcal{L}^{(s)}$ is defined as the Laplacian of $D^{(s)}$. 
Laplacians of Eulerian directed graphs

- Chung[2005] gave the definition of general directed graphs.
Laplacians of Eulerian directed graphs

- Chung [2005] gave the definition of general directed graphs.
- $D$ is a Eulerian directed graph.
Laplacians of Eulerian directed graphs

- Chung[2005] gave the definition of general directed graphs.
- $D$ is a Eulerian directed graph.
- $A$ is the adjacency matrix of $D$, i.e., $A(u,v) = 1$ if $(u,v) \in E(D)$ and 0 otherwise.
Chung[2005] gave the definition of general directed graphs.

$D$ is a Eulerian directed graph.

$A$ is the adjacency matrix of $D$, i.e., $A(u,v) = 1$ if $(u,v) \in E(D)$ and 0 otherwise.

$T$ is a diagonal matrix such that $T(v,v) = d_v$. 

The Laplacian $L = I - T - \frac{1}{2}(A + A^T)T - \frac{1}{2}(A - A^T)$. 

Laplacians of Eulerian directed graphs
Chung[2005] gave the definition of general directed graphs.

$D$ is a Eulerian directed graph.

$A$ is the adjacency matrix of $D$, i.e., $A(u, v) = 1$ if $(u, v) \in E(D)$ and 0 otherwise.

$T$ is a diagonal matrix such that $T(v, v) = d_v$.

The Laplacian $L = I - T^{-1/2}(\frac{A+A'}{2})T^{-1/2}$. 
For $1 \leq s \leq r/2$, $\mathcal{L}^{(s)}$ is the Laplacian of the weighted graph $G^{(s)}$. 
For $1 \leq s \leq r/2$, $\mathcal{L}^{(s)}$ is the Laplacian of the weighted graph $G^{(s)}$.

For $r/2s \leq r - 1$, $\mathcal{L}^{(s)}$ is the Laplacian of the Eulerian directed graph $D^{(s)}$. 
For $1 \leq s \leq r - 1$, the eigenvalues of $\mathcal{L}^{(s)}$ are listed as

$$0 = \lambda_0^{(s)} \leq \lambda_1^{(s)} \leq \cdots \leq \lambda_{\text{max}}^{(s)}.$$
For $1 \leq s \leq r - 1$, the eigenvalues of $\mathcal{L}^{(s)}$ are listed as

$$0 = \lambda_{0}^{(s)} \leq \lambda_{1}^{(s)} \leq \cdots \leq \lambda_{\text{max}}^{(s)}.$$  

Let $\bar{\lambda}^{(s)} = \max\{|1 - \lambda_{1}^{(s)}|, |1 - \lambda_{\text{max}}^{(s)}|\}$. 
For $1 \leq s \leq r/2$ and $H = K^r_n$, we have

\[
\begin{array}{|c|c|c|}
\hline
\lambda_1^{(s)} & \lambda_{\text{max}}^{(s)} & \lambda^{(s)} \\
1 - \frac{s(s-1)}{(n-s)(n-s-1)} & 1 + \frac{s}{n-s} & \frac{s}{n-s} \\
\hline
\end{array}
\]
Eigenvalues of small complete hypergraphs

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\lambda_1^{(4)}$</th>
<th>$\lambda_1^{(3)}$</th>
<th>$\lambda_1^{(2)}$</th>
<th>$\lambda_1^{(1)}$</th>
<th>$\lambda_{\text{max}}^{(1)}$</th>
<th>$\lambda_{\text{max}}^{(2)}$</th>
<th>$\lambda_{\text{max}}^{(3)}$</th>
<th>$\lambda_{\text{max}}^{(4)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_6^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6/5</td>
<td>3/2</td>
</tr>
<tr>
<td>$K_7^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7/10</td>
<td>7/6</td>
<td>7/6</td>
<td>3/2</td>
</tr>
<tr>
<td>$K_6^4$</td>
<td>1/3</td>
<td></td>
<td></td>
<td></td>
<td>5/6</td>
<td>6/5</td>
<td>6/5</td>
<td>3/2</td>
</tr>
<tr>
<td>$K_7^4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3/8</td>
<td>9/10</td>
<td>7/6</td>
<td>7/5</td>
</tr>
<tr>
<td>$K_6^5$</td>
<td>0.1464</td>
<td>1/2</td>
<td>5/6</td>
<td>6/5</td>
<td>6/5</td>
<td>3/2</td>
<td>3/2</td>
<td>1.76759</td>
</tr>
<tr>
<td>$K_7^5$</td>
<td>0.1977</td>
<td>5/8</td>
<td>9/10</td>
<td>7/6</td>
<td>7/6</td>
<td>7/5</td>
<td>3/2</td>
<td>1.809</td>
</tr>
</tbody>
</table>
Different $s$-th Laplacians are related

**Theorem (Lu and Peng 2011)**

Suppose that $H$ is an $r$-uniform hypergraph. We have

\[
\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \ldots \geq \lambda_1^{\lfloor r/2 \rfloor},
\]

\[
\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \ldots \leq \lambda_{\max}^{\lfloor r/2 \rfloor}.
\]
Application to the mixing rate of random walks

- $H$ is an $s$-connected $r$-uniform hypergraph.
Application to the mixing rate of random walks

- $H$ is an $s$-connected $r$-uniform hypergraph.
- For $0 \leq \alpha < 1$, $f_k$ is the joint distribution of the $k$-th stop of the $\alpha$-lazy random $s$-walk.
Application to the mixing rate of random walks

- $H$ is an $s$-connected $r$-uniform hypergraph.
- For $0 \leq \alpha < 1$, $f_k$ is the joint distribution of the $k$-th stop of the $\alpha$-lazy random $s$-walk.
- $\pi$ is the stationary distribution, $\pi(x) = d_x/\text{vol}(G^{(s)})$. 
Application to the mixing rate of random walks

- $H$ is an $s$-connected $r$-uniform hypergraph.
- For $0 \leq \alpha < 1$, $f_k$ is the joint distribution of the $k$-th stop of the $\alpha$-lazy random $s$-walk.
- $\pi$ is the stationary distribution, $\pi(x) = d_x/vol(G^{(s)})$.

**Theorem (Lu and Peng 2011)**

For $1 \leq s \leq r/2$, we have

$$\| (f_k - \pi) T^{-1/2} \| \leq (\bar{\lambda}_\alpha^{(s)})^k \| (f_0 - \pi) T^{-1/2} \|.$$

Here $\bar{\lambda}_\alpha^{(s)} = \max\{|1 - (1 - \alpha)\lambda_1^{(s)}|, |1 - (1 - \alpha)\lambda_{\text{max}}^{(s)}|\}$. 
Application to the mixing rate of random walks

- $H$ is an $s$-connected hypergraph.
Application to the mixing rate of random walks

- $H$ is an $s$-connected hypergraph.
- $0 < \alpha < 1$, the joint distribution $f_k$ at the $k$-th stop of the $\alpha$-lazy random $s$-walk at time $k$. 

Theorem (Lu and Peng 2011)

For $r/2 < s \leq r - 1$, we have

$$\| (f_k - \pi)_T - \frac{1}{2} \| \leq (\sigma(s) \alpha)^k \| (f_0 - \pi)_T - \frac{1}{2} \|.$$ 

Here, $(\sigma(s) \alpha)^2 \leq \alpha^2 + 2\alpha(1 - \alpha)\lambda(s) + (1 - \alpha)^2$. 
Application to the mixing rate of random walks

- $H$ is an $s$-connected hypergraph.
- $0 < \alpha < 1$, the joint distribution $f_k$ at the $k$-th stop of the $\alpha$-lazy random $s$-walk at time $k$.
- $\pi$ is the stationary distribution, $\pi(x) = d_x/\text{vol}(D^{(s)})$. 
Application to the mixing rate of random walks

- $H$ is an $s$-connected hypergraph.
- $0 < \alpha < 1$, the joint distribution $f_k$ at the $k$-th stop of the $\alpha$-lazy random $s$-walk at time $k$.
- $\pi$ is the stationary distribution, $\pi(x) = d_x / \text{vol}(D^{(s)})$.

**Theorem (Lu and Peng 2011)**

For $r/2 < s \leq r - 1$, we have

$$

\|(f_k - \pi)T^{-1/2}\| \leq (\sigma^{(s)}_{\alpha})^k \|(f_0 - \pi)T^{-1/2}\|.

$$

Here, $(\sigma^{(s)}_{\alpha})^2 \leq \alpha^2 + 2\alpha(1 - \alpha)\lambda_1^{(s)} + (1 - \alpha)^2$. 

For $1 \leq s \leq r/2$ and $x, y \in V^s$, the $s$-distance $d^{(s)}(x, y)$ is the distance of $x$ and $y$ in $G^{(s)}$. 

Application to the $s$-distance and $s$-diameter

For $1 \leq s \leq r/2$ and $x, y \in V^s$, the $s$-distance $d^{(s)}(x, y)$ is the distance of $x$ and $y$ in $G^{(s)}$.

$X, Y \subseteq V^s, d^{(s)}(X, Y) = \min\{d^{(s)}(x, y) \mid x \in X, y \in Y\}$. 
Application to the $s$-distance and $s$-diameter

- For $1 \leq s \leq r/2$ and $x, y \in V^s$, the $s$-distance $d^{(s)}(x, y)$ is the distance of $x$ and $y$ in $G^{(s)}$.
- $X, Y \subseteq V^s$, $d^{(s)}(X, Y) = \min\{d^{(s)}(x, y) \mid x \in X, y \in Y\}$.
- The $s$-th diameter of $H$, denoted by $\text{diam}^{(s)}(H)$, is the diameter of the graph $G^{(s)}$. 
Application to the $s$-distance and $s$-diameter

**Theorem (Lu and Peng 2011)**

Suppose $H$ is an $r$-uniform hypergraph. For integer $s$ such that $1 \leq s \leq \frac{r}{2}$, we have

$$d^{(s)}(X, Y) \leq \log \sqrt{\frac{\text{vol}(X)\text{vol}(Y)}{\text{vol}(X)\text{vol}(Y)}} \left[ \log \frac{\lambda_{\text{max}}^{(s)} + \lambda_{1}^{(s)}}{\lambda_{\text{max}}^{(s)} - \lambda_{1}^{(s)}} \right].$$
Application to the $s$-distance and $s$-diameter

**Theorem (Lu and Peng 2011)**

Suppose $H$ is an $r$-uniform hypergraph. For integer $s$ such that $1 \leq s \leq \frac{r}{2}$, the $s$-diameter of an $r$-uniform hypergraph $H$ satisfies

$$diam^{(s)}(H) \leq \left[ \log \frac{\log \lambda^{(s)}_{\max} + \lambda^{(s)}_1}{\log \lambda^{(s)}_{\max} - \lambda^{(s)}_1} \right].$$
For $r/2 < s \leq r - 1$ and $x, y \in V^s$, we define $s$-th distance $d^{(s)}(x, y)$ to be the distance of $x$ and $y$ in the directed graph $D^{(s)}$. 
Application to the $s$-distance and $s$-diameter

- For $r/2 < s \leq r - 1$ and $x, y \in V^s$, we define $s$-th distance $d^{(s)}(x, y)$ to be the distance of $x$ and $y$ in the directed graph $D^{(s)}$.
- $X, Y \subseteq V^s$, $d^{(s)}(X, Y) = \min\{d^{(s)}(x, y) \mid x \in X, y \in Y\}$. 
For $r/2 < s \leq r - 1$ and $x, y \in V^s$, we define $s$-th distance $d^{(s)}(x, y)$ to be the distance of $x$ and $y$ in the directed graph $D^{(s)}$.

$X, Y \subseteq V^s, d^{(s)}(X, Y) = \min\{d^{(s)}(x, y) \mid x \in X, y \in Y\}$.

The $s$-th diameter of $H$, denoted by $\text{diam}^{(s)}(H)$, is the diameter of the directed graph $D^{(s)}$. 

Application to the $s$-distance and $s$-diameter
Let $H$ be an $r$-uniform hypergraph. For $r/2 < s \leq r - 1$ and $X, Y \subseteq V^s$, if $H$ is $s$-connected, then we have

$$d^{(s)}(X, Y) \leq \left\lfloor \log \frac{\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(X)\text{vol}(Y)} \right\rfloor + 1.$$
Theorem (Lu and Peng 2011)

For \( r/2 < s \leq r - 1 \), suppose that an \( r \)-uniform hypergraph \( H \) is \( s \)-connected. The \( s \)-diameter of \( H \) satisfies

\[
diam^{(s)}(H) \leq \left[ \frac{2 \log \frac{\text{vol}(V^s)}{\delta^{(s)}}}{\log \frac{2}{2 - \lambda_1^{(s)}}} \right].
\]
Application to edge expansions

For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} dx$. 
Application to edge expansions

- For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.
- $\text{vol} \left( \binom{V}{s} \right) = |E(H)| \binom{r}{s}$.
Application to edge expansions

- For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.
- $\text{vol} \left( \binom{V}{s} \right) = |E(H)| \binom{r}{s}$.
- The density $e(S)$ of $S$ is $\frac{\text{vol}(S)}{\text{vol}(\binom{V}{s})}$.
For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.

$\text{vol}\left(\binom{V}{s}\right) = |E(H)|\binom{r}{s}$.

The density $e(S)$ of $S$ is $\frac{\text{vol}(S)}{\text{vol}\left(\binom{V}{s}\right)}$.

$\bar{S}$ is the complement of $S$ in $\binom{V}{s}$. $e(\bar{S}) = 1 - e(S)$.
Application to edge expansions

- For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.
- $\text{vol} \left( \binom{V}{s} \right) = |E(H)| \binom{r}{s}$.
- The density $e(S)$ of $S$ is $\frac{\text{vol}(S)}{\text{vol}(\binom{V}{s})}$.
- $\bar{S}$ is the complement of $S$ in $\binom{V}{s}$. $e(\bar{S}) = 1 - e(S)$.
- For $1 \leq t \leq s \leq r - t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $E(S, T) = \{ F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \emptyset, x \cup y \subseteq F \}$.
Application to edge expansions

- For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.
- $\text{vol}\left(\binom{V}{s}\right) = |E(H)|\binom{r}{s}$.
- The density $e(S)$ of $S$ is $\frac{\text{vol}(S)}{\text{vol}(\binom{V}{s})}$.
- $\bar{S}$ is the complement of $S$ in $\binom{V}{s}$. $e(\bar{S}) = 1 - e(S)$.
- For $1 \leq t \leq s \leq r - t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $E(S, T) = \{F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \emptyset, x \cup y \subseteq F\}$.
- $|E(\binom{V}{s}, \binom{V}{t})| = |E(H)|\frac{r!}{s!t!(r-s-t)!}$. 
Application to edge expansions

- For $S \subseteq \binom{V}{s}$, $\text{vol}(S)$ is $\sum_{x \in S} d_x$.
- $\text{vol}(\binom{V}{s}) = |E(H)| \binom{r}{s}$.
- The density $e(S)$ of $S$ is $\frac{\text{vol}(S)}{\text{vol}(\binom{V}{s})}$.
- $\bar{S}$ is the complement of $S$ in $\binom{V}{s}$. $e(\bar{S}) = 1 - e(S)$.
- For $1 \leq t \leq s \leq r - t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let $E(S, T) = \{F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \emptyset, x \cup y \subseteq F\}$.
- $|E(\binom{V}{s}, \binom{V}{t})| = |E(H)| \frac{r!}{s!t!(r-s-t)!}$.
- Let $e(S, T) = \frac{|E(S, T)|}{|E(\binom{V}{s}, \binom{V}{t})|}$. 
Application to edge expansion

Theorem (Lu and Peng 2011)

For $1 \leq t \leq s \leq \frac{r}{2}$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$. We have

$$|e(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(S')e(T')}.$$
Application to edge expansion

**Theorem (Lu and Peng 2011)**

For $1 \leq t < \frac{r}{2} < s < s + t \leq r$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$. If $|x \cap y| \neq \min\{t, 2s - r\}$ for any $x \in S$ and $y \in T$, then we have

$$\left| \frac{1}{2} e(S, T) - e(S)e(T) \right| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$
Application to edge expansion

- For $\frac{r}{2} < s \leq r - 1$ and $S, T \subseteq \binom{V}{s}$.
Application to edge expansion

- For $\frac{r}{2} < s \leq r - 1$ and $S, T \subseteq \binom{V}{s}$.
- $E'(S, T) = \{ F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y \}$. 
Application to edge expansion

- For $\frac{r}{2} < s \leq r - 1$ and $S, T \subseteq \binom{V}{s}$.
- $E'(S, T) = \{F \in E(H) | \exists x \in S, \exists y \in T, F = x \cup y\}$.
- $|E'(\binom{V}{s}, \binom{V}{s})| = |E(H)| \frac{r!}{(r-s)!(2s-r)!(r-s)!}$. 
Application to edge expansion

- For \( \frac{r}{2} < s \leq r - 1 \) and \( S, T \subseteq \binom{V}{s} \).
- \( E'(S, T) = \{ F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y \} \).
- \(|E'(\binom{V}{s}, \binom{V}{s})| = |E(H)| \frac{r!}{(r-s)!(2s-r)!(r-s)!} \).
- \( e'(S, T) = \frac{|E'(S,T)|}{|E'(\binom{V}{s},\binom{V}{s})|} \).
Application to edge expansion

Theorem (Lu and Peng 2011)

For $\frac{r}{2} < s \leq r - 1$ and $S, T \subseteq \binom{V}{s}$, we have

$$|e'(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}.$$
Open problems

- What is the phase transition of random uniform hypergraphs?
Open problems

- What is the phase transition of random uniform hypergraphs?
- When $r/2 < s \leq r - 1$, what is the eigenvalues of the $s$-th Laplacian $\mathcal{L}^{(s)}$ for complete hypergraphs?
Open problems

- What is the phase transition of random uniform hypergraphs?
- When $r/2 < s \leq r - 1$, what is the eigenvalues of the $s$-th Laplacian $\mathcal{L}^{(s)}$ for complete hypergraphs?
- Is $\lambda_1^{(s)}$ related to quasirandom hypergraphs?
Thank you