1. Let \( \{x_i\}_{i=1}^n \) be i.i.d. sample from a strictly positive density that is symmetric about \( \theta \), show that the Huber \( M \)-estimator for location is consistent for \( \theta \).

Solution: Huber \( M \)-estimator \( \hat{\mu}_n \) is the solution of
\[
\Psi_n(\mu) := \frac{1}{n} \sum_{i=1}^n \psi(X_i - \mu) = 0,
\]
where
\[
\psi(x) = \begin{cases} 
-\tau & \text{if } x < -\tau \\
x & \text{if } -\tau \leq x \leq \tau \\
\tau & \text{if } x > \tau 
\end{cases}
\]
Define
\[
\Psi(\mu) := \mathbb{E}[\psi(X - \mu)],
\]
then by weak law of large numbers, \( \Psi_n(\mu) \xrightarrow{P} \Psi(\mu) \) for every \( \mu \). Moreover, \( \Psi_n(\mu) \) is nonincreasing in \( \mu \), and it remains to check that for any \( \epsilon > 0 \),
\[
\Psi(\theta - \epsilon) > 0 > \Psi(\theta + \epsilon).
\]
For the first inequality,
\[
\Psi(\theta - \epsilon) = \int_{-\infty}^{\infty} \psi(x - (\theta - \epsilon)) f(x) dx = \int_{-\infty}^{\infty} \psi(x) f(x + (\theta - \epsilon)) dx = \int_{-\infty}^{\infty} \psi(x) g(x) dx,
\]
where \( g(x) \) is symmetric about \( \epsilon \). Then
\[
\int_{-\infty}^{\infty} \psi(x) g(x) dx > \int_{-\infty}^{0} \psi(x) g(x) dx + \int_{2\epsilon}^{\infty} \psi(x) g(x) dx = \int_{-\infty}^{0} [\psi(2\epsilon - x) + \psi(x)] g(x) dx \geq 0.
\]
The second inequality follows similarly. Together, they guarantee that \( \hat{\mu}_n \xrightarrow{P} \theta \).

2. Let \( \{x_i\}_{i=1}^n \) be i.i.d. sample from a strictly positive density. Define
\[
\psi(x) = \frac{2}{1 + e^{-x}} - 1,
\]
and \( \hat{\theta}_n \) be the solution of
\[
\sum_{i=1}^n \psi(X_i - \theta) = 0.
\]
(a) Show that \( \hat{\theta}_n \xrightarrow{P} \theta_0 \) for some \( \theta_0 \), and express \( \theta_0 \) in the density of observations;

Solution: Notice that \( \psi(x) = \tanh(x/2) \), and if we define
\[
\Psi(\theta) := \mathbb{E}[\psi(X - \theta)] = \int_{-\infty}^{\infty} \tanh((x - \theta)/2) f(x) dx,
\]
then $\Psi(\theta)$ is strictly decreasing with $\Psi(-\infty) = 1$ and $\Psi(\infty) = -1$, so there is a unique solution of $\Psi(\theta) = 0$, and we denote the solution as $\theta_0$. If we further define

$$\Psi_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \psi(X_i - \theta),$$

then we have $\Psi_n(\theta) \overset{p}{\to} \Psi(\theta)$ and $\Psi(\theta - \epsilon) > 0 > \Psi(\theta + \epsilon)$ for every $\epsilon > 0$, which implies $\hat{\theta}_n \overset{p}{\to} \theta_0$. (b) Show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution and find the limit variance.

**Solution:** It can be shown that $\psi(x)$ is twice continuously differentiable with

$$\psi'(x) = \tanh'(x/2) = \frac{\text{sech}^2(x/2)}{2}$$

and

$$\psi''(x) = \tanh''(x/2) = -\frac{\text{sech}(t/2) \tanh(t/2)}{2}.$$  

Furthermore, we have $|\psi'(x)| \leq 1/2$, $|\psi''(x)| \leq 1/2$, and $\psi'(x) > 0$. Together, they give us

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \overset{d}{\to} N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{4\mathbb{E}[\tanh^2((X - \theta)/2)]}{(\mathbb{E}[\text{sech}^2((X - \theta)/2)])^2}.$$

3. Let $\{x_i\}_{i=1}^{n}$ be i.i.d. sample from Uniform$(0, 1)$, determine the relative efficiency of the sample median and the sample mean.

**Solution:** For $X \sim \text{Uniform}(0, 1)$, we have $\mathbb{E}[X] = 1/2$ and $\text{var}(X) = 1/12$, so the asymptotic variance of sample mean is $1/12$. Besides, the population median is also $1/2$ and the asymptotic variance of sample median is $1/(4f^2(0)) = 1/4$. Hence, the relative efficiency is $1/3$.

4. Let $\{x_i\}_{i=1}^{n}$ be i.i.d. sample from $N(\theta, 1)$, find the relative efficiency of the Huber estimator and the sample mean.

**Solution:** The asymptotic variance of sample mean is 1, and the asymptotic variance of Huber estimator is $\mathbb{E}[\psi_\theta']^2/(\mathbb{E}[\psi_\theta'])^2$. Then we compute these two items,

$$\mathbb{E}[\psi_\theta'] = 2 \int_0^{\tau} x^2 \phi(x) \, dx + 2\tau^2 \int_{\tau}^{\infty} \phi(x) \, dx,$$

and

$$\mathbb{E}[\psi_\theta''] = \int_{-\tau}^{\tau} \phi(x) \, dx,$$

where $\phi(x)$ denotes the density of standard normal distribution. So the relative efficiency is

$$\frac{\left(\int_{-\tau}^{\tau} \phi(x) \, dx\right)^2}{2 \int_0^{\tau} x^2 \phi(x) \, dx + 2\tau^2 \int_{\tau}^{\infty} \phi(x) \, dx}.$$