Math 281C Homework 1

Due: Apr 17, 5 pm

1. Let $U \in \mathbb{R}$ be a random variable such that $\mathbb{E}[U] = 0$ and $a \leq U \leq b$ almost surely, for some constants $b \geq a$. Prove that for any $\lambda \geq 0$,

$$
\Psi_U(\lambda) := \log \mathbb{E}e^{\lambda U} \leq (b - a)^2 \lambda^2 / 8.
$$

2. Let $Z \sim \mathcal{N}(0,1)$. Prove that for any $t > 0$,

$$
\frac{t}{\sqrt{2\pi(1 + t^2)}} e^{-t^2/2} \leq \mathbb{P}(Z \geq t) \leq \frac{1}{\sqrt{2\pi t}} e^{-t^2/2}.
$$

3. Consider the function

$$
h(u) = (1 + u) \log(1 + u) - u,
$$

where $u > -1$. Prove the for any $u \geq 0$,

$$
h(u) \geq \frac{u^2}{2(1 + u/3)}.
$$

4. Assume $\{\xi_i, \mathcal{F}_i\}_{i=1}^n$ is a martingale difference sequence, where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_n$ are $\sigma$-fields. That is, each $\xi_i$ is $\mathcal{F}_i$-measurable and $\mathbb{E}[\xi_i | \mathcal{F}_{i-1}] = 0$ almost surely. Moreover, the conditional distribution of $\xi_n$ given $\mathcal{F}_{n-1}$ is supported on an interval with width bounded by $R_n$. Show that

$$
\mathbb{E}[e^{\lambda \xi_n} | \mathcal{F}_{n-1}] \leq e^{\lambda^2 R_n^2 / 8}.
$$