PROBLEM 1. (Problem 1.9.1.2).

Solution. Since \( X_1, \ldots, X_n \) are uncorrelated \( \sim (\theta, \sigma^2) \), we have \( \mathbb{E}[X_i^2] = \text{Var}(X_i) + \mathbb{E}[X_i]^2 = \sigma^2 + \theta^2 \) for all \( i \) and \( \mathbb{E}[X_iX_j] = \theta^2 \) for all \( i \neq j \). Let \( Y = \sum_i \alpha_i X_i \) where \( \sum_i \alpha_i = 1 \). Then clearly \( \mathbb{E}[Y] = \theta \), and

\[
\mathbb{E}[Y^2] = \sum_i \alpha_i^2 \mathbb{E}[X_i^2] + \sum_{i \neq j} \alpha_i \alpha_j \mathbb{E}[X_iX_j]
= \theta^2 + \sigma^2 \sum_i \alpha_i^2
\geq \theta^2 + \sigma^2 \frac{1}{n} \left( \sum_i \alpha_i \right)^2 = \theta^2 + \frac{\sigma^2}{n},
\]

where Cauchy-Schwarz inequality was used in the last inequality, so with equality if and only if \( \alpha_1 = \ldots = \alpha_n = \frac{1}{n} \). Thus,

\[
\text{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \geq \frac{\sigma^2}{n}.
\]

Thus, the mean has the smallest variance. \( \square \)

PROBLEM 2. (Problem 1.9.1.3).

Solution of (a). We have \( \mathbb{E}[X_i^2] = \frac{\sigma^2}{\beta_i} + \theta^2 \) for each \( i \) and \( \mathbb{E}[X_iX_j] = \theta^2 \) for all \( i \neq j \). Thus, \( \mathbb{E}[Y_i] = \theta \) for all \( i \), and

\[
\mathbb{E}[Y^2] = \sum_i \alpha_i^2 \mathbb{E}[X_i^2] + \sum_{i \neq j} \alpha_i \alpha_j \mathbb{E}[X_iX_j]
= \theta^2 + \sigma^2 \sum_i \frac{\alpha_i^2}{\beta_i}
\geq \theta^2 + \sigma^2 \frac{\left( \sum_i \alpha_i \right)^2}{\sum_i \beta_i} = \theta^2 + \frac{\sigma^2}{\sum_i \beta_i},
\]
with equality if and only if $\frac{\beta_1}{\alpha_1} = \ldots = \frac{\beta_n}{\alpha_n} = \text{const.}$, which implies for each $i$, $\alpha_i = \beta_i / \sum_i \beta_i$. Thus,

$$\text{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \geq \frac{\sigma^2}{\sum_{i=1}^n \beta_i},$$

and the minimum is achieved by the estimator $Y = \sum_i \frac{\beta_i}{\sum_j \beta_j} X_i$. \qed

**Solution of (b).** From the conditions given, $\mathbb{E}[X_i] = \theta^2 + \sigma^2$ for all $i$, and $\mathbb{E}[X_i X_j] = \text{Cov}(X_i, X_j) + \mathbb{E}[X_i] \mathbb{E}[X_j] = \theta^2 + \rho \sigma^2$. Thus,

$$\mathbb{E}[Y^2] = \sum_i \alpha_i^2 \mathbb{E}[X_i^2] + \sum_{i \neq j} \alpha_i \alpha_j \mathbb{E}[X_i X_j]$$

$$= \sum_i \alpha_i^2 (\sigma^2 + \theta^2) + \sum_{i \neq j} \alpha_i \alpha_j (\theta^2 + \rho \sigma^2)$$

$$= \theta^2 + \rho \sigma^2 + \sigma^2 (1 - \rho) \sum_i \alpha_i^2$$

$$\geq \theta^2 + \sigma^2 \left(\frac{1}{n} + \frac{n-1}{n} \rho\right).$$

Like in the previous problems, we use Cauchy-Schwarz inequality, and we have equality achieved if and only if $\alpha_1 = \ldots = \alpha_n = \frac{1}{n}$. Finally we have

$$\text{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \geq \sigma^2 \left(\frac{1}{n} + \frac{n-1}{n} \rho\right),$$

and this minimum can be only achieved by the mean $\overline{X}$. \qed

**Problem 3.** (Problem 1.9.1.4). Assuming $X \sim (\theta, \sigma^2)$, $Y \sim (\theta, \tau^2)$, and $\text{corr}(X, Y) = \rho$, we have $\mathbb{E}[XY] = \rho \sigma \tau + \theta^2$.

**Solution of (a).**

$$\text{Var}\left(\frac{X + Y}{2}\right) = \mathbb{E}\left[\left(\frac{X + Y}{2}\right)^2\right] - \left(\mathbb{E}\left[\frac{X + Y}{2}\right]\right)^2$$

$$= \frac{1}{4} \mathbb{E}[X^2 + Y^2] + \frac{1}{2} \mathbb{E}[XY] - \theta^2$$

$$= \frac{1}{4} (2\theta^2 + \sigma^2 + \tau^2) + \frac{1}{2} (\rho \sigma \tau - \theta^2)$$

$$= \frac{1}{4} (\sigma^2 + \tau^2 + 2\rho \sigma \tau)$$

$$> \sigma^2 = \text{Var}(X)$$

is the equivalent condition of $\text{Var}(X) < \text{Var}\left(\frac{(X + Y)}{2}\right)$.
Solution of (b).

\[ f(\alpha) = \text{Var}(\alpha X + (1 - \alpha)Y) \]
\[ = \mathbb{E}[(\alpha X + (1 - \alpha)Y)^2] - (\mathbb{E}[\alpha X + (1 - \alpha)Y])^2 \]
\[ = \alpha^2(\theta^2 + \sigma^2) + (1 - \alpha)^2(\theta^2 + \tau^2) + 2\alpha(1 - \alpha)(\rho\sigma\tau + \theta^2) - \theta^2 \]
\[ = \alpha^2(\sigma^2 + \tau^2 - 2\rho\sigma\tau) + 2\alpha(-\tau^2 + \rho\sigma\tau) + \tau^2. \]

Thus, the minimizer of \( f(\alpha) \) is

\[ \alpha_\ast = \frac{\tau(\tau - \rho\sigma)}{\sigma^2 + \tau^2 - 2\rho\sigma\tau}. \]

To make this minimizer negative, we require the equivalent condition \( \tau < \rho\sigma \).

\[ \square \]

**Problem 4.** (Problem 1.9.1.7).

Solution of (a). \( P(X \leq m) = 1 - P(X > m) \geq \frac{1}{2} \iff P(X > m) \leq \frac{1}{2} \), and \( P(X \geq m) = 1 - P(X < m) \geq \frac{1}{2} \iff P(X < m) \leq \frac{1}{2} \).

Solution of (b). Let \( m_0 = \inf\{m : P(X \leq m) \geq \frac{1}{2}\} \) and \( m_1 = \sup\{m : P(X \geq m) \geq \frac{1}{2}\} \). We now only need to prove that \( m_0 \) and \( m_1 \) are medians. By definition, for any \( \epsilon > 0 \), we have \( P(X \leq m_0 - \epsilon) < \frac{1}{2} \), which implies \( P(X > m_0 - \epsilon) > \frac{1}{2} \). By taking \( \epsilon \to 0 \), \( P(X \geq m_0) \geq \frac{1}{2} \), and thus \( m_0 \) is a median. We can prove that \( m_1 \) is median by the same reasoning.

\[ \square \]

**Problem 5.** (Problem 1.9.1.8).

Solution. Let \( m \in [m_0, m_1] \) is a median. Note that \( \phi(a) = \mathbb{E}|X - a| = \mathbb{E}[(X - a)(1 - 21_{\{X \leq m\}})] \) for any \( a \). Therefore, for \( c > m_1 \) we have

\[ \mathbb{E}|X - c| - \mathbb{E}|X - m| = \mathbb{E}[(X - c)(1 - 21_{\{X < c\}})] - \mathbb{E}[(X - m)(1 - 21_{\{X \leq m\}})] \]
\[ = \mathbb{E}[(X - c)((1 - 21_{\{X < c\}}) - (1 - 21_{\{X \leq m\}}))] - \mathbb{E}[(c - m)(1_{\{X > m\}} - 1_{\{X \leq m\}})] \]
\[ = 2\mathbb{E}[(1_{\{X < c\}} - 1_{\{X \leq m\}})(c - X)] + (c - m)\mathbb{E}1_{\{X \leq m\}} - 1_{\{X > m\}} \]
\[ = (c - m)(P(X \leq m) - P(X > m)) + 2\int_{m < x < c} (c - x)dP(x) \]
\[ \geq 0, \]

because \( P(X \leq m) \geq \frac{1}{2} \), and \( P(X > m) \leq \frac{1}{2} \), and this implies that \( \mathbb{E}|X - c| \geq \mathbb{E}|X - m| \) for any \( c > m_1 \). We can prove that this holds for \( c < m_0 \) by the same reasoning. Hence, any median minimizes \( \phi(a) \).

\[ \square \]

**Problem 6.** (Problem 1.9.1.9).

Solution of (a). Obvious from definition.

Solution of (b). If we assume uniform distribution over the finite set \( \{x_1, \ldots, x_n\} \), then \( \sum_i |x_i - a| = n\phi(a) \) using the notation in Problem 1.9.1.8. Hence, any median is a minimizer of the sum of absolute deviations.

\[ \square \]
Solution of (c). Without loss of generality, we can assume that \( x_i \neq 0 \) for all \( i \). (If \( x_i = 0 \), this would make the term \( |y_i - bx_i| = |y_i| \), a constant.) Then,

\[
\arg\min_b \sum_i |y_i - bx_i| = \arg\min_b \sum_i \frac{|x_i|}{\sum_j |x_j|} |y_i - b| = \arg\min_b \mathbb{E}[|Z - b|],
\]

where we let \( Z \) be a random variable taking values from \( \{y_1/x_1, \ldots, y_n/x_n\} \), and let pmf of \( Z \) be \((|x_1|, \ldots, |x_n|)/(\sum_j |x_j|))\). Therefore, a minimizer \( b \) is any median of the random variable \( Z \).

\[\text{Problem 7. (Problem 1.9.1.10).}\]

Solution. Note that

\[
\arg\min_a \sum_{i=1}^n (h(x_i) - h(a))^2 = \arg\min_a \left( h(a)^2 - \frac{2}{n} h(a) \sum_{i=1}^n h(x_i) \right) = h^{-1}\left( \frac{1}{n} \sum_{i=1}^n h(x_i) \right),
\]

since \( h(\cdot) \) is a monotone function. One can easily check that 1) \( h(x) = x \) gives AM, 2) \( h(x) = \log x \) gives GM, and 3) \( h(x) = \frac{1}{x} \) gives HM.

\[\text{Problem 8. (Problem 1.9.1.12).}\]

Solution of (a). Let \( f_k(x) = \frac{k-1}{2} \frac{1}{(1+|x|)^k} \). \( f_k \) is a density function because

\[
\int_{-\infty}^{\infty} f_k(x) dx = \frac{k-1}{2} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^k} = (k-1) \int_0^{\infty} (1+x)^{-k} dx = (k-1) \left( -\frac{(1+x)^{-(k-1)}}{k-1} \right)_0^\infty = 1.
\]

Now assume \( 1 \leq l < k - 1 \). For even \( l \),

\[
\mathbb{E}[X^l] = \int_{-\infty}^{\infty} x^l f_k(x) dx = \int_{-\infty}^{\infty} x^l k-1 \frac{dx}{2(1+|x|)^k} = (k-1) \int_0^{\infty} x^l (1+x)^{-k} dx = (k-1) \left( -\frac{x^l (1+x)^{-(k-1)}}{k-1} \right)_0^\infty + \frac{l}{k-1} \int_0^{\infty} x^{l-1} (1+x)^{-(k-1)} dx
\]

\[
= (k-1) \frac{l}{k-1} \int_0^{\infty} x^{l-1} (1+x)^{-(k-1)} dx = \ldots
\]
\[
\frac{l!(k-l-1)!}{(k-2)!} \int_0^\infty (1 + x)^{-(k-l)}dx
\]

\[
< \infty,
\]

since \( k - l > 1 \). For odd \( l \), since \( \mathbb{E}[^{|X|^l}] < \infty \) by the above result, we have \( \mathbb{E}[X^l] = 0 \) by symmetry.

**Solution of (b).** Note that

\[
2 \int f^2(x)dx = \frac{(k-1)^2}{2} \int_{-\infty}^\infty \frac{dx}{(1 + |x|)2k} = (k-1)^2 \int_0^\infty (1 + x)^{-2k}dx = \frac{(k-1)^2}{2k-1} < \frac{k-1}{2} = f(0),
\]

because \( 2(k-1) < 2k-1 \) for all \( k \).

**Problem 9.** (Problem 1.9.4.1).

**Solution.** Recall: A scale family of distributions has densities of the form \( f(x; \sigma) = \frac{1}{\sigma} \psi \left( \frac{x}{\sigma} \right) \) where \( \sigma > 0 \) and a location family of distributions has densities of the form \( f(x; \mu) = \psi(x - \mu) \) where \( -\infty < \mu < \infty \). Let \( Y = \log X \), and let \( f_X(x; \sigma) = \frac{1}{\sigma} \psi \left( \frac{x}{\sigma} \right) \) be the pdf of \( X \). Then we have \( F_Y(y) = F_X(e^y) \), and thus \( f_Y(y) = e^y f_X(e^y) = e^{y-\log \sigma} \psi(e^{y-\log \sigma}) = \varphi(y - \log \sigma) \) for some function \( \varphi(\cdot) \). This proves that \( Y = \log X \) is a location family.

**Problem 10.** (Problem 1.9.4.2).

**Solution.** We want to prove that if \( X \sim U(0, \theta) \), then \( -\log X \sim \text{exp} \). The cdf of \( X \) is \( F_X(x) = \frac{x}{\theta} \) for \( x \in [0, \theta] \). Let \( Y = -\log X \in [-\log \theta, \infty) \). Then, \( F_Y(y) = \mathbb{P}(-\log X \leq y) = \mathbb{P}(X \geq e^{-y}) = 1 - F_X(e^{-y}) = 1 - e^{-y}/\theta \) for \( y \in [-\log \theta, \infty) \). Hence, we get the pdf of \( Y \) \( f_Y(y) = e^{-y}/\theta \) for \( y \in [-\log \theta, \infty) \), which implies that \( Y \sim E(-\log \theta, 1) \).

**Problem 11.** (Problem 1.9.4.13(b)).

**Solution.** We have \( U \sim E(0, 1) \), \( X = bu^{1/c} \), where \( b, c > 0 \). Then the cdf of \( X \) becomes \( F_X(x) = \mathbb{P}(bu^{1/c} \leq x) = \mathbb{P}(Y \leq (x/b)^c) = 1 - e^{-(x/b)^c} \), which implies that the pdf of \( X \) is \( f_X(x) = \frac{c}{\theta} (\frac{x}{\theta})^{c-1} e^{-(\frac{x}{\theta})^c} \), for \( x > 0 \).

**Problem 12.** (Additional problem 1). Suppose \( X_1, \ldots, X_n \sim \text{i.i.d.} \), and \( \mathbb{E}X_1^3 < \infty \). Then prove \( sk(X) = sk(X_1)/\sqrt{n} \) and \( ku(X) = ku(X_1)/n \).

**Solution of (a).**

\[
\mu_3 = \mathbb{E}[(X - \mu)^3] = \mathbb{E} \left[ \frac{1}{n} \sum_i (X_i - \mu)^3 \right]
\]

\[
= \frac{1}{n^3} \mathbb{E} \left[ \sum_i (X_i - \mu)^3 \right] = \frac{n}{n^3} \mathbb{E}(X_1 - \mu)^3 = \frac{\mathbb{E}(X_i - \mu)^3}{n^2}
\]
Therefore,

\[ sk(X) = \frac{\mu_3}{(\text{var}(X))^{3/2}} = \frac{E(X_1 - \mu)^3}{n^2 \cdot (E(X_1 - \mu)^2)^{3/2}} = \frac{1}{\sqrt{n}} \frac{E(X_1 - \mu)^3}{[E(X_1 - \mu)^2]^{3/2}} = \frac{1}{\sqrt{n}} sk(X_1). \]

As \( n \to \infty \), \( sk(X) \to 0 \). \( \square \)

**Solution of (b).**

\[ \mu_4 = E[(\bar{X} - \mu)^4] = E\left[\frac{1}{n} \sum_i (X_i - \mu)\right]^4 \]

\[ = \frac{1}{n^4} \left\{ \sum_i E(X_i - \mu)^4 + \sum_i \sum_{i<j} \sum_{i<j} \sum_{i<j} 6 E(X_i - \mu)^2 E(X_j - \mu)^2 \right\} \]

\[ = \frac{1}{n^4} \left\{ n E(X_1 - \mu)^4 + 3n(n-1)[E(X_1 - \mu)^2]^2 \right\} \]

Therefore,

\[ ku(X) = \frac{E[(\bar{X} - \mu)^4]}{\text{var}(X)^2} - 3 \]

\[ = \frac{1}{n^4} \left\{ n E(X_1 - \mu)^4 + 3n(n-1)[E(X_1 - \mu)^2]^2 \right\} - 3 \]

\[ = \frac{1}{n^4} \left\{ \frac{E(X_1 - \mu)^4}{[\text{var}(X_1)^2] - 3} \right\} \]

\[ = \frac{1}{n} ku(X_1). \]

As \( n \to \infty \), \( ku(X) \to 0 \). \( \square \)

**Problem 13.** *(Additional problem 2).* Show that the usually-used estimator of \( \sigma^2 \) is always unbiased.

**Solution of (13).* We assume \( X_1, \ldots, X_n \sim \text{i.i.d.} (\mu, \sigma^2) \). Then for all \( i \) \( E[X_i^2] = \mu^2 + \sigma^2 \) and \( E[\bar{X}^2] = \mu^2 + \frac{\sigma^2}{n} \). Now we have, for each \( i \),

\[ E(X_i - \bar{X})^2 = E X_i^2 + E \bar{X}^2 - 2 E X_i \bar{X} \]

\[ = \mu^2 + \frac{\sigma^2}{n} + \frac{\sigma^2}{n} - 2 \mu^2 \]

\[ = \frac{\sigma^2}{n}. \]
\[
(\mu^2 + \sigma^2) + \left(\mu^2 + \frac{\sigma^2}{n}\right) - \frac{2}{n}(\mu^2 + \sigma^2) + (n - 1)\mu^2
\]
\[= (1 - \frac{1}{n})\sigma^2.\]

Hence, by linearity of expectation,
\[
E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = (n - 1)\sigma^2. \tag{\textbullet}
\]

**Problem 14.** *(Additional problem 3).* What is the variance of the sample variance (the estimator in problem 13)?

*Solution of (14).* Google “variance of the sample variance”, you’ll find the best answer comes in here “http://math.stackexchange.com/questions/72975/variance-of-sample-variance”. In the future, use Google to help yourself learn Statistics and whatever you want! \tag{\textbullet}