Problem 1. (Problem 2.7.1.5).

Solution of (a). Using Cauchy-Schwarz inequality, we have

$$(E_{XY})^2 = \left( \int xy \, d\mu \right)^2 \leq \left( \int x^2 \, d\mu \right) \left( \int y^2 \, d\mu \right) = E_{X^2}E_{Y^2},$$

where $\mu$ is the measure on $\mathbb{R}^2$ for random variables $X$ and $Y$ respectively. Therefore, if we plug in $X \leftarrow X - E_X$ and $Y \leftarrow Y - E_Y$ into this inequality, we have

$$\text{Cov}(X,Y)^2 \leq \text{Var}(X) \text{Var}(Y).$$

Solution of (b). From the equality condition of Cauchy-Schwarz inequality, we have the equality in the variance inequality if and only if $X - E_X = c(Y - E_Y)$ with probability 1, for some constant $c$, and thus for some constant $a$ and $b$,

$$P(X = aY + b) = 1.$$

Problem 2. (Problem 2.7.1.6).

Solution. For any $\lambda$, we have

$$\int (f + \lambda g)^2 \, dP = \int f^2 \, dP + 2\lambda \int fg \, dP + \lambda^2 \int g^2 \, dP \geq 0.$$

Therefore, the quadratic equation of $\lambda$ should have at most one root, so the determinant of this equation should be $\leq 0$. That is,

$$\left( \int fg \, dP \right)^2 \leq \left( \int f^2 \, dP \right) \left( \int g^2 \, dP \right).$$

Problem 3. (Problem 2.7.1.8).

Solution. We have $\text{Var}(\delta), \text{Var}(\delta') < \infty$. Then, using Cauchy-Schwarz inequality, it follows that

$$\text{Var}(\delta' - \delta) = | \text{Var}(\delta' - \delta) |$$

$$= | \text{Var}(\delta) + \text{Var}(\delta') - 2 \text{Cov}(\delta', \delta) |$$

$$\leq \text{Var}(\delta) + \text{Var}(\delta') + 2 | \text{Cov}(\delta', \delta) |$$

$$\leq \text{Var}(\delta) + \text{Var}(\delta') + 2 \sqrt{\text{Var}(\delta) \text{Var}(\delta')} < \infty.$$
**Problem 4.** (Problem 2.7.1.12).

*Solution.* Suppose there exists two UMVUE \( \delta_1 \) and \( \delta_2' \) of \( g(\theta) \). Then \( \text{Var}(\delta_1) = \text{Var}(\delta_2) \) by definition. Using the fact that \( \delta_1 \) and \( \delta_2 \) have the minimum variance among the unbiased estimators, consider a new unbiased estimator \( \frac{\delta_1 + \delta_2}{2} \):

\[
\text{Var}(\delta_1) \leq \text{Var}\left(\frac{\delta_1 + \delta_2}{2}\right) = \frac{1}{4} \left( \text{Var}(\delta_1) + \text{Var}(\delta_2) + 2 \text{Cov}(\delta_1, \delta_2) \right)
\leq \frac{1}{2} \left( \text{Var}(\delta_1) + \sqrt{\text{Var}(\delta_1) \text{Var}(\delta_2)} \right)
= \text{Var}(\delta_1).
\]

Here in the second inequality, we use Cauchy-Schwarz inequality. Since the lower bound and the upper bound agree, we have the equality condition for Cauchy-Schwarz inequality, which implies \( \mathbb{P}(\delta_2 = a\delta_1 + b) = 1 \) for some constants \( a \) and \( b \). From unbiasedness, however, we should have \( a = 1 \) and \( b = 0 \). Thus, \( \mathbb{P}(\delta_1 = \delta_2) = 1 \). In other words, the UMVUE is unique. \( \square \)

**Problem 5.** (Additional Problem). Suppose we sample \( X_1, \ldots, X_n \) from \( \mathcal{N}(\mu, \sigma^2) \) and the target function is \( g(\theta) = \sigma^2 \). Compare the risk functions of two estimators \( \delta_1(X) = \frac{1}{n-1} \sum_i (X_i - \overline{X})^2 \) and \( \delta_2(X) = \frac{1}{n} \sum_i (X_i - \overline{X})^2 \).

*Solution.* We will assume that the loss function is given as a quadratic loss: \( L(\theta, d) = (g(\theta) - d)^2 \). Note that \( \delta_2(X) \) is an unbiased estimator of \( \sigma^2 \). We will use the following formula:

\[
R_\delta(\theta) = \text{Var}_\theta(\delta(X)) + \text{bias}^2(g(\theta)).
\]

First, using the variance formula of \( \delta_1(X) \), which is an unbiased estimator of variance, from Homework 1, we have

\[
R_{\delta_1}(\theta) = \text{Var}_\theta(\delta_1(X))
= \frac{n\mu^4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}
= \frac{2\sigma^4}{n-1}.
\]

On the other hand,

\[
R_{\delta_2}(\theta) = \text{Var}_\theta(\delta_2(X)) + \text{bias}^2(g(\theta))
= \frac{(n-1)^2}{n^2} \text{Var}_\theta(\delta_1(X)) + (\mathbb{E} \delta_2(X) - \sigma^2)^2
= \frac{(n-1)^2}{n^2} \frac{2\sigma^4}{n-1} + \left( \frac{n-1}{n} \sigma^2 - \sigma^2 \right)^2
= \frac{(2n-1)\sigma^4}{n^2}.
\]

One can easily check that \( R_{\delta_2}(\theta) < R_{\delta_1}(\theta) \), since \( 2n^2 > (n-1)(2n-1) \). This implies that though \( \delta_2 \) is not an unbiased estimator, \( \delta_2 \) shows better performance than an unbiased estimator \( \delta_1 \) in terms of risk function. \( \square \)