MATH 281A: Homework #2

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Problem 1. *(Problem 2.7.1.5).*
Solution of (a). Using Cauchy-Schwarz inequality, we have

$$(\mathsf{E}\,XY)^2 = \left(\int xyd\mu\right)^2 \leq \left(\int x^2 d\mu\right)\left(\int y^2 d\mu\right) = \mathsf{E}\,X^2\,\mathsf{E}\,Y^2,$$

where μ is the measure on $\mathbb{R}^2$ for random variables X and Y respectively. Therefore, if we plug in $X \leftarrow X - \mathsf{E}\,X$ and $Y \leftarrow Y - \mathsf{E}\,Y$ into this inequality, we have

$$\operatorname{Cov}(X,Y)^2 \leq \operatorname{Var}(X)\operatorname{Var}(Y). \qquad \square$$

Solution of (b). From the equality condition of Cauchy-Schwarz inequality, we have the equality in the variance inequality if and only if $X - \mathsf{E}\,X = c(Y - \mathsf{E}\,Y)$ with probability 1, for some constant c, and thus for some constant a and b,

$$\mathsf{P}(X = aY + b) = 1. \qquad \square$$

Problem 2. *(Problem 2.7.1.6).*
Solution. For any λ, we have

$$\int (f+\lambda g)^2 d\,\mathsf{P} = \int f^2 d\,\mathsf{P} + 2\lambda \int fg d\,\mathsf{P} + \lambda^2 \int g^2 d\,\mathsf{P} \geq 0.$$

Therefore, the quadratic equation of λ should have at most one root, so the determinant of this equation should be ≤ 0. That is,

$$\left(\int fg d\,\mathsf{P}\right)^2 \leq \left(\int f^2 d\,\mathsf{P}\right)\left(\int g^2 d\,\mathsf{P}\right). \qquad \square$$

Problem 3. *(Problem 2.7.1.8).*

Solution. We have $\operatorname{Var}(\delta), \operatorname{Var}(\delta') < \infty$. Then, using Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\operatorname{Var}(\delta' - \delta) &= |\operatorname{Var}(\delta' - \delta)| \\
&= |\operatorname{Var}(\delta) + \operatorname{Var}(\delta') - 2\operatorname{Cov}(\delta', \delta)| \\
&\leq \operatorname{Var}(\delta) + \operatorname{Var}(\delta') + 2|\operatorname{Cov}(\delta', \delta)| \\
&\leq \operatorname{Var}(\delta) + \operatorname{Var}(\delta') + 2\sqrt{\operatorname{Var}(\delta)\operatorname{Var}(\delta')} < \infty. \qquad \square
\end{aligned}$$

Problem 4. *(Problem 2.7.1.12).*

Solution. Suppose there exists two UMVUE δ_1 and δ_2' of $g(\theta)$. Then $\mathrm{Var}(\delta_1) = \mathrm{Var}(\delta_2)$ by definition. Using the fact that δ_1 and δ_2 have the minimum variance among the unbiased estimators, consider a new unbiased estimator $\frac{\delta_1+\delta_2}{2}$:

$$\begin{aligned}\mathrm{Var}(\delta_1) \le \mathrm{Var}\left(\frac{\delta_1+\delta_2}{2}\right) &= \frac{1}{4}\left(\mathrm{Var}(\delta_1) + \mathrm{Var}(\delta_2) + 2\,\mathrm{Cov}(\delta_1,\delta_2)\right)\\ &\le \frac{1}{2}\left(\mathrm{Var}(\delta_1) + \sqrt{\mathrm{Var}(\delta_1)\,\mathrm{Var}(\delta_2)}\right)\\ &= \mathrm{Var}(\delta_1).\end{aligned}$$

Here in the second inequality, we use Cauchy-Schwarz inequality. Since the lower bound and the upper bound agree, we have the equality condition for Cauchy-Schwarz inequality, which implies $\mathsf{P}(\delta_2 = a\delta_1 + b) = 1$ for some constants a and b. From unbiasedness, however, we should have $a = 1$ and $b = 0$. Thus, $\mathsf{P}(\delta_1 = \delta_2) = 1$. In other words, the UMVUE is unique. □

Problem 5. *(Additional Problem).* Suppose we sample $X_1, \ldots, X_n$ from $\mathcal{N}(\mu, \sigma^2)$ and the target function is $g(\theta) = \sigma^2$. Compare the risk functions of two estimators $\delta_1(X) = \frac{1}{n-1}\sum_i (X_i - \overline{X})^2$ and $\delta_2(X) = \frac{1}{n}\sum_i (X_i - \overline{X})^2$.

Solution. We will assume that the loss function is given as a quadratic loss: $L(\theta, d) = (g(\theta) - d)^2$. Note that $\delta_2(X)$ is an unbiased estimator of σ^2. We will use the following formula:

$$R_\delta(\theta) = \mathrm{Var}_\theta(\delta(X)) + \mathrm{bias}^2(g(\theta)).$$

First, using the variance formula of $\delta_1(X)$, which is an unbiased estimator of variance, from Homework 1, we have

$$\begin{aligned}R_{\delta_1}(\theta) &= \mathrm{Var}_\theta(\delta_1(X))\\ &= \frac{\mu_4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}\\ &= \frac{2\sigma^4}{n-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}R_{\delta_2}(\theta) &= \mathrm{Var}_\theta(\delta_2(X)) + \mathrm{bias}^2(g(\theta))\\ &= \frac{(n-1)^2}{n^2}\,\mathrm{Var}_\theta(\delta_1(X)) + (\mathsf{E}\,\delta_2(X) - \sigma^2)^2\\ &= \frac{(n-1)^2}{n^2}\frac{2\sigma^4}{n-1} + \left(\frac{n-1}{n}\sigma^2 - \sigma^2\right)^2\\ &= \frac{(2n-1)\sigma^4}{n^2}.\end{aligned}$$

One can easily check that $R_{\delta_2}(\theta) < R_{\delta_1}(\theta)$, since $2n^2 > (n-1)(2n-1)$. This implies that though δ_2 is not an unbiased estimator, δ_2 shows better performance than an unbiased estimator δ_1 in terms of risk function. □