1. \( E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) = \text{Var}(X) \), provided \( E[X^2] < \infty \).

Proof. We calculate

\[
\text{Var}(E[X|Y]) = E[(E[X|Y])^2] - (E[E[X|Y]])^2
\]
and

\[
\text{Var}(X|Y) = E[(X - E[X|Y])^2|Y]
\]

where the last line follows from the fact that \( E[X|Y] \) is measurable with respect to the \( \sigma \)-field generated by \( Y \). Taking expectation we have

\[
E[\text{Var}(X|Y)] = E[E[X^2|Y]] - 2E[E[XE[X|Y]|Y]] + E[E[(E[X|Y])^2]]
\]
\[
= E[X^2] - 2E[XE[X|Y]] + E[(E[X|Y])^2]
\]

And so

\[
E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) = E[X^2] - 2E[XE[X|Y]] + 2E[(E[X|Y])^2] - (E[X])^2.
\]

Now we use the result that if \( Z \) is a random variable measurable with respect to the \( \sigma \)-field \( \mathcal{F} \), and \( E[|W||E[|ZW|]] < \infty \), then \( E[ZW|\mathcal{F}] = ZE[W|\mathcal{F}] \) (see Durrett, *Probability Theory and Examples*, page 228). Since \( E[X|Y] \) is always measurable with respect to \( \sigma(Y) \), we then have that

\[
\]
so that

\[
E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]) = E[X^2] - (E[X])^2 = \text{Var}(X).
\]

2. If \( X, Y \sim_{iid} U(0, 1) \), then the density of \( \frac{X}{Y} \) is given by

\[
f_{\frac{X}{Y}}(z) = \begin{cases} 
\frac{1}{2} & \text{if } z \leq 1, \\
\frac{1}{2z^2} & \text{if } z > 1.
\end{cases}
\]

Proof. We note that since \( X \) is uniform on \( (0, 1) \) its cumulative distribution is given by

\[
F_X(x) = \begin{cases} 
x & \text{if } 0 < x \leq 1, \\
1 & \text{if } x > 1
\end{cases}
\]
and similarly with $F_Y(y)$. We have

$$P \left( \frac{X}{Y} \leq z \right) = P(X \leq zY) = E[1_{X \leq zY}] = E[E[1_{X \leq zY} | Y]] = E[P(X \leq zY | Y)] = E[F_X(zY)] = E[zY 1_{zY \leq 1} + 1_{zY > 1}] = zE[Y 1_{Y \leq \frac{1}{z}}] + P(Y > \frac{1}{z}) = z \int_0^1 y 1_{y \leq \frac{1}{z}} dy + 1 - P(Y \leq \frac{1}{z}) = z \left( \min \left\{ 1, \frac{1}{z} \right\} \right)^2 + 1 - \min \left\{ 1, \frac{1}{z} \right\}$$

so that

$$F_{\frac{X}{Y}}(z) = \begin{cases} \frac{z}{2} & \text{if } z \leq 1, \\ 1 - \frac{1}{2z} & \text{if } z > 1. \end{cases}$$

Differentiating gives the desired result.

3. If $P(X = 0) < 1$, $Y$ is independent of $X$, and $\lim_{z \to 0} f_{|Y|}(z) > 0$, then $E[\frac{|X|}{Y^r}] = \infty$.

Proof. Since $X$ and $Y$ are independent, $E[\frac{|X|}{Y^r}] = E[|X||E[\frac{1}{|Y|^r}]]$. Since $P(X = 0) < 1$, $0 < E[|X|] \leq \infty$. Thus it suffices to show that $E[\frac{1}{|Y|^r}] = \infty$. For all $\epsilon > 0$,

$$E \left[ \frac{1}{|Y|^r} \right] = \int_0^\infty \frac{1}{|y|} f_{|Y|}(|y|) dy \geq \int_0^\epsilon \frac{1}{|y|} f_{|Y|}(|y|) dy$$

so letting $\epsilon \to 0$, we get the point mass at 0 of the integrand on the right hand side, which we identify as

$$\frac{1}{|y|} f_{|Y|}(|y|) 1_{|y| = 0} = \lim_{|y| \to 0} \frac{1}{|y|} f_{|Y|}(|y|) = \infty$$

since $\lim_{|y| \to 0} f_{|Y|}(z) > 0$.

4. Let $E(t|X|) = (E[|X|^t])^{1/t}$. $E(t|X|)$ is monotonically increasing in $t$, for $t > 0$, and $\lim_{t \to 0} E(t|X|) = e^{E(\log|X|)}$.

Proof. For two random variables $Y$ and $Z$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, Hölder’s inequality says

$$E[|YZ|] \leq (E[|Y|^p])^{1/p}(E[|Z|^q])^{1/q}.$$  

Let $s > t$. Then if we let $p = s/t$ and $q = s/(s-t)$, we have that $p, q > 1$, so applying Hölder’s inequality to the random variables $|X|^t$ and $1_\Omega$, where $\Omega$ is our entire sample space,

$$E[|X|^t] = E[|X|^t 1_\Omega] \leq (E[|X|^p])^{t/p}(E[1_\Omega]^s/(s-t)) = (E[|X|^s])^{t/s}.$$  

It follows that $(E[|X|^t])^{1/t} \leq (E[|X|^s])^{1/s}$, so $E(t|X|)$ is monotonically increasing. Now, if we rewrite $|X|^t = e^{t \log |X|}$, then since $\varphi(x) = e^x$ is a convex function, Jensen’s inequality gives

$$(E[|X|^t])^{1/t} = (E[e^{t \log |X|}])^{1/t} \geq (e^{E[t \log |X|]})^{1/t} = e^{E(\log |X|)}.$$
for all $t > 0$. Since $E^{(t)}[X]$ is increasing and bounded from below, it certainly has a limit as $t \to 0$. This limit is the same as the limit of $E^{(t/t)}[X]$ as $t \to \infty$. Now consider $\log E^{(1/t)}[X] = \log(E[|X|^{1/t}])^t$. We use the fact that for $a > 0$, and any $n > 0$, we have the inequality

$$\log a \leq n(a^{1/n} - 1).$$

So, if $a = E^{(1/t)}[X]$ and $n = t$,

$$\log(E[|X|^{1/t}])^t \leq t((E[|X|^{1/t}])^t - 1) = t(E[|X|^{1/t}] - 1) = E\left[\frac{|X|^{1/t} - 1}{1/t}\right].$$

For large enough $t$, the integrand above is dominated by $\frac{1}{p}(|X|^p - 1)$ for some $p$ (Since we must assume that $E^{(t)}[X]$ is finite for some positive $t$), so when we take the limit $t \to \infty$, we can pull it inside the expectation. By L'Hôpital's rule

$$\lim_{n \to 0} \frac{a^n - 1}{n} = \lim_{n \to 0} a^n \log a = \log a.$$

Putting a bunch of things together, we have

$$\lim_{t \to \infty} \log(E[|X|^{1/t}])^t \leq \lim_{t \to \infty} E\left[\frac{|X|^{1/t} - 1}{1/t}\right] = E\left[\lim_{t \to \infty} \frac{|X|^{1/t} - 1}{1/t}\right] = E[\log|X|].$$

Finally this gives us

$$e^{E[\log|X|]} \leq \lim_{t \to 0} (E[|X|^t])^{1/t} \leq e^{E[\log|X|]}.$$

Nice proof!