PROBLEM 1. (Problem 2.7.1.22).
Solution of (a). $X \sim \text{Bern}(p)$, where $\frac{1}{4} < p < \frac{3}{4}$.

$$
\delta^*(X) = \begin{cases} 
Y_0 \sim \text{Unif} \left( -\frac{1}{2}, \frac{1}{2} \right) & \text{if } X = 0 \\
Y_1 \sim \text{Unif} \left( \frac{1}{2}, \frac{3}{2} \right) & \text{if } X = 1.
\end{cases}
$$

Hence,

$$
E[\delta^*(X)] = E[\delta^*(X) | X = 0]P(X = 0) + E[\delta^*(X) | X = 1]P(X = 1) = E[Y_0]q + E[Y_1]p = p,
$$
so $\delta^*(X)$ is unbiased for $p$. \hfill \Box

Solution of (b). The loss function is given as $L(p, d) = \mathbb{1}\{|d - p| \geq \frac{1}{4}\}$. The risk function for $\delta^*(X)$ is,

$$
R_{\delta^*(X)}(p) = E_p[L(p, \delta^*(X))]
= E_p[\mathbb{1}\{|\delta^*(X) - p| \geq \frac{1}{4}\}]
= P\left(|\delta^*(X) - p| \geq \frac{1}{4}\right)
= P\left(|Y_0 - p| \geq \frac{1}{4}\right)q + P\left(|Y_1 - p| \geq \frac{1}{4}\right)p
= q\left(P\left(Y_0 \geq p + \frac{1}{4}\right) + P\left(Y_0 \leq p - \frac{1}{4}\right)\right) + p\left(P\left(Y_1 \geq p + \frac{1}{4}\right) + P\left(Y_1 \leq p - \frac{1}{4}\right)\right)
= q\left(p - \frac{1}{4} - \left(-\frac{1}{2}\right)\right) + p\left(\frac{3}{2} - \left(p + \frac{1}{4}\right)\right)
= \frac{1}{4} + 2pq = 2p(1 - p) + \frac{1}{4}.
$$

On the other hand,

$$
R_X(p) = E_p[L(p, X)] = E_p[\mathbb{1}\{|X - p| \geq \frac{1}{4}\}]
$$

1
\[
\begin{align*}
= \mathbb{P}\left(|X - p| \geq \frac{1}{4}\right) &= \mathbb{P}\left(X \geq p + \frac{1}{4}\right) + \mathbb{P}\left(X \leq p - \frac{1}{4}\right) = p + q = 1.
\end{align*}
\]

(Alternatively, since \(X\) can only take values of 0 or 1, with \(\frac{1}{4} < p < \frac{3}{4}\), \(|X - p|\) is greater than \(\frac{1}{4}\) with probability 1, thus the risk of estimating \(p\) by \(X\) equals 1.)

One can easily check that \(R_\delta(p) > R_X(p)\) for all \(p \in \left(\frac{1}{4}, \frac{3}{4}\right)\). \(\square\)

**Problem 2.** (Problem 2.7.2.15).

*Solution.* Noting that
\[
S_{XY} = \sum_i (X_i - \bar{X})(Y_i - \bar{Y}) = \sum_i X_i Y_i - n \bar{X} \bar{Y},
\]
we have
\[
\mathbb{E}[S_{XY}] = n \mathbb{E}[X_1 Y_1] - \frac{1}{n} \sum_i \sum_j \mathbb{E}[X_i Y_j]
\]
\[
= n \mathbb{E}[X_1 Y_1] - \frac{1}{n} \left( n \mathbb{E}[X_1 Y_1] + n(n - 1) \mathbb{E}[X_1] \mathbb{E}[Y_1] \right)
\]
\[
= (n - 1) \mathbb{E}[X_1 Y_1] - (n - 1) \mathbb{E}[X_1] \mathbb{E}[Y_1]
\]
\[
= (n - 1) \text{Cov}(X_1, Y_1).
\]

Hence, \(\frac{S_{XY}}{n-1}\) is an unbiased estimator for \(\text{Cov}(X, Y)\). \(\square\)

**Problem 3.** (Problem 2.7.5.3).

<table>
<thead>
<tr>
<th>Table 5.1. (I[\tau(\theta)]) for Some Exponential Families</th>
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</thead>
<tbody>
<tr>
<td>Distribution</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>(N(\xi, \sigma^2))</td>
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<tr>
<td>(N(\xi, \sigma^2))</td>
</tr>
<tr>
<td>(b(p, n))</td>
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<tr>
<td>(P(\lambda))</td>
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<tr>
<td>(\Gamma(\alpha, \beta))</td>
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</tbody>
</table>

*Solution of (a).* \(X \sim N(\xi, \sigma^2)\), \(i(\xi) = \frac{1}{\sigma^2}\). Since
\[
\log f_{\xi, \sigma^2}(x) = C - \frac{1}{2\sigma^2}(x - \xi)^2 - \frac{1}{2} \log \sigma^2,
\]
we have
\[
\frac{\partial^2}{\partial \xi^2} \log f_{\xi, \sigma^2}(x) = -\frac{1}{\sigma^2}.
\]
Hence,
\[ i(\xi) = -\mathbb{E} \left[ \frac{\partial^2}{\partial \xi^2} \log f_{\xi, \sigma^2}(X) \right] = \frac{1}{\sigma^2}. \]

Solution of (b). As we derived in the previous problem,
\[ \frac{\partial^2}{\partial (\sigma^2)^2} \log f_{\xi, \sigma^2}(x) = -\frac{1}{\sigma^6} (x - \xi)^2 + \frac{1}{2\sigma^4}. \]

Hence,
\[ i(\sigma^2) = \frac{1}{\sigma^6} \mathbb{E}[(X - \xi)^2] - \frac{1}{2\sigma^4} = \frac{1}{2\sigma^4}. \]

Solution of (c). \( X \sim \text{Binom}(n, p) \). Then \( p(k) = \binom{n}{k} p^k (1 - p)^{n-k} \), and \( \log p(k) = C + k \log p + (n - k) \log(1 - p) \). Hence,
\[ \frac{\partial}{\partial p} \log p(k) = \frac{k - n - k}{1 - p}, \quad \text{and} \quad \frac{\partial^2}{\partial p^2} \log p(k) = -\frac{k - n - k}{(1 - p)^2}. \]

Therefore,
\[ i(p) = \frac{1}{p^2} \mathbb{E}[X] + \frac{1}{(1 - p)^2} \mathbb{E}[n - X] = \frac{np}{p^2} + \frac{n(1 - p)}{(1 - p)^2} = \frac{n}{p} + \frac{n}{1 - p} = \frac{n}{p(1 - p)}. \]

Solution of (d). \( X \sim \text{Poisson}(\lambda) \). Then \( p(x) = e^{-\lambda x} / x! \), and \( \log p(x) = -\lambda + x \log \lambda - \log x! \). Hence,
\[ \frac{\partial^2}{\partial \lambda^2} \log p(x) = \frac{x}{\lambda^2}, \]

and thus
\[ i(\lambda) = \frac{1}{\lambda^2} \mathbb{E}[X] = \frac{1}{\lambda}. \]

Solution of (e). \( X \sim \text{Gamma}(\alpha, \beta) \). Then \( f_{\alpha, \beta}(x) = \frac{\beta^{-\alpha} e^{-x/\beta} x^{\alpha-1}}{\Gamma(\alpha)} \) for \( x \geq 0 \), and \( \log f_{\alpha, \beta}(x) = -\alpha \log \beta - \log \Gamma(\alpha) - \frac{x}{\beta} + (\alpha - 1) \log x \). Hence,
\[ \frac{\partial^2}{\partial \lambda^2} \log f_{\alpha, \beta}(x) = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}. \]

Note that
\[ \mathbb{E}_{\alpha, \beta}[X] = \alpha \beta \int_0^\infty \frac{\beta^{-(\alpha+1)} e^{-x/\beta} x^{\alpha-1}}{\Gamma(\alpha+1)} dx = \alpha \beta. \]

Therefore,
\[ i(\beta) = \frac{2}{\beta^3} \mathbb{E}_{\alpha, \beta}[X] - \frac{\alpha}{\beta^2} = \frac{2}{\beta^3} \alpha \beta - \frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2}. \]

**Problem 4.** (Problem 2.7.5.4).
Furthermore, by chain rule, we have for the binomial distribution Binom(n,p)

\[ i(\sigma) = i(\sigma^2), \text{ however it is wrong!} \]

For \( \mathcal{N}(0,\sigma^2) \), \( i(\sigma^2) = \frac{1}{\sigma^2} \). By equation (5.11) \( I^*(\xi) = I[h(\xi) : h'(\xi)]^2 \) in chapter 2, with \( h(\cdot) \) being we have \( i(\sigma) = i(\sigma^2) \cdot (2\sigma)^2 = 2 \sigma^2 \).

(Understand and memorize equation (5.11) and it will make your life easier during 281 exams.)

**Problem 5.** (Problem 2.7.5.5).

_Solution._ \( X \sim \text{Geom}(p) \). Then \( p_p(k) = pq^{k-1} \) for \( k = 1, 2, \ldots \), where \( q = 1 - p \), and \( \log p_p(k) = \log p + (k-1) \log(1 - p) \). Hence,

\[
\frac{\partial^2}{\partial p^2} \log p_p(k) = -\frac{1}{p^2} - \frac{k-1}{(1-p)^2}.
\]

Therefore, the Fisher information of \( p \) is

\[
i(p) = \frac{1}{p^2} + \mathbb{E}[X] - 1 = \frac{1}{p^2} + \frac{1/p - 1}{(1-p)^2} = \frac{1}{p^2(1-p)}.
\]

**Problem 6.** (Problem 2.7.5.6).

_Solution._ If \( X \sim \text{Poisson}(\lambda) \), we know that \( \log p_\lambda(x) = -\lambda + x \log \lambda - \log x! \). Then

\[
\frac{\partial}{\partial \sqrt{\lambda}} \log p_\lambda(x) = -2\sqrt{\lambda} + \frac{2x}{\sqrt{\lambda}}, \text{ and } \frac{\partial^2}{\partial (\sqrt{\lambda})^2} \log p_\lambda(x) = -2 - \frac{2x}{\lambda}.
\]

Hence,

\[
i(\sqrt{\lambda}) = 2 + \frac{2}{\lambda} \mathbb{E}[X] = 4.
\]

**Problem 7.** (Problem 2.7.5.8(b)).

_Solution._ We want to find a function \( g(p) \) for which the amount of information is independent of \( p \) for the binomial distribution Binom(n,p). We know that \( \log p_p(x) = C + x \log p + (n-x) \log(1-p) \), and

\[
\frac{\partial}{\partial p} \log p_p(x) = \frac{x}{p} - \frac{n-x}{1-p} = \frac{X-np}{p(1-p)}.
\]

Furthermore, by chain rule, we have

\[
\left( \frac{\partial}{\partial g(p)} \log p_p(X) \right)^2 = \left( \frac{1}{g'(p)} \frac{\partial}{\partial g} \log p_p(X) \right)^2 = \frac{(X-np)^2}{g'(p)^2p^2(1-p)^2}.
\]

Hence,

\[
i(g(p)) = \mathbb{E} \left[ \left( \frac{\partial}{\partial g(p)} \log p_p(X) \right)^2 \right] = \frac{\text{Var}(X)}{g'(p)^2p^2(1-p)^2} = \frac{n}{g'(p)^2p^2(1-p)^2},
\]

and if \( g'(p) \sqrt{p(1-p)} \) is independent of \( p \), then so is \( i(g(p)) \). After some following algebra,

\[
\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2dx}{\sqrt{(1-(1-2x))(1+(1-2x))}} \quad (\cos \theta := 1-2x)
\]
\[
\theta + C = 2 \arcsin(\sqrt{x}) + C,
\]
you can verify that a function of the form
\[
g(p) = 2a \arcsin(\sqrt{p}) + b
\]
for any constants \(a\) and \(b\) makes \(i(g(p))\) independent of \(p\). \(\square\)

**Problem 8.** *(Problem 2.7.5.16).*

**Solution of (a).** Assume that we are given a scale family
\[
\mathcal{F} = \left\{ \frac{1}{|\theta|} f \left( \frac{x}{\theta} \right) : \theta > 0 \right\},
\]
where \(f\) is a density function. We want to verify the following formula for \(i(\theta)\), where \(\theta\) is a scale parameter.
\[
i(\theta) = \frac{1}{\theta^2} \int \left( \frac{y f'(y)}{f(y)} + 1 \right)^2 f(y) dy.
\]
Let \(f_\theta(x) = \frac{1}{\theta} f \left( \frac{x}{\theta} \right)\). Then, \(\log f_\theta(x) = \log f \left( \frac{x}{\theta} \right) - \log \theta\). Hence,
\[
\frac{\partial}{\partial \theta} \log f_\theta(x) = \frac{1}{f \left( \frac{x}{\theta} \right)} \frac{\partial}{\partial \theta} f \left( \frac{x}{\theta} \right) - \frac{1}{\theta} = \frac{1}{f \left( \frac{x}{\theta} \right)} \left( -\frac{x}{\theta^2} \right) f' \left( \frac{x}{\theta} \right) - \frac{1}{\theta} = -\frac{1}{\theta} \left( \frac{x f' \left( \frac{x}{\theta} \right)}{\frac{x}{\theta} f \left( \frac{x}{\theta} \right)} + 1 \right).
\]

Therefore,
\[
i(\theta) = \mathbb{E} \left[ \left( \frac{\partial}{\partial \theta} \log f_\theta(X) \right)^2 \right] = \frac{1}{\theta^2} \int \left( \frac{x f' \left( \frac{x}{\theta} \right)}{\frac{x}{\theta} f \left( \frac{x}{\theta} \right)} + 1 \right)^2 \frac{1}{\theta} f \left( \frac{x}{\theta} \right) dx
\]
\[
= \frac{1}{\theta^2} \int \left( \frac{y f'(y)}{f(y)} + 1 \right)^2 f(y) dy.
\]
\(\square\)

**Solution of (b).** Let \(\xi = \log \theta\). Then the information of \(\xi\) is
\[
i(\xi) = \mathbb{E} \left[ \left( \frac{\partial}{\partial \xi} \log f_\theta(X) \right)^2 \right] = \mathbb{E} \left[ \left( \frac{1}{\frac{\partial}{\partial \theta} \log f_\theta(X)} \right)^2 \right] = \mathbb{E} \left[ \theta^2 \left( \frac{\partial}{\partial \theta} \log f_\theta(X) \right)^2 \right] = \theta^2 i(\theta),
\]
and from the formula we derived above \(\theta^2 i(\theta)\) is independent of \(\theta\). *(Alternatively, you could make use of equation (5.11). You will end up with the same result.)* \(\square\)

**Solution of (c).** If \(X \sim C(0, \theta)\), then \(f_\theta(x) = \frac{\theta}{\pi \theta^2 + x^2}\), and \(f(x) = \frac{1}{\pi(1 + x^2)}\). Since \(f'(x) = \frac{-2x}{\pi(1 + x^2)^2}\), it follows that \(\frac{x f'(x)}{f(x)} + 1 = \frac{1 - x^2}{1 + x^2}\), and the formula gives us
\[
i(\theta) = \frac{1}{\theta^2} \int_{-\infty}^{\infty} \frac{\left(1 - x^2\right)^2}{\pi (1 + x^2)^2} \frac{dx}{\pi (1 + x^2)} = \frac{1}{\pi \theta^2} \int_{-\infty}^{\infty} \frac{(1 - x^2)^2}{(1 + x^2)^3} dx = \frac{1}{2\theta^2}.
\]
Here we use the following integration:

\[ \int_{-\infty}^{\infty} \frac{(1 - x^2)^2}{(1 + x^2)^3} \, dx = \int_{-\pi/2}^{\pi/2} \frac{(1 - \tan^2 \theta)^2}{\sec^6 \theta} \sec^2 \theta \, d\theta \]

\[ = \pi - \int_{-\pi/2}^{\pi/2} \sin^2 2\theta \, d\theta = \frac{\pi}{2}. \]

**Problem 9.** *(Additional Problem 1).*

_Solution._ Refer to Problem 2.7.5.16 (a).

**Problem 10.** *(Additional Problem 2).* Prove that if \( X \sim \text{geom}(p) \), then \( \mathbb{E}[X] = \frac{1}{p} \).

_Solution._ For \( k = 1, 2, \ldots \), \( P(X = k) = pq^{k-1} \). Hence,

\[ \mathbb{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = p \frac{d}{dq} \left( \sum_{k=1}^{\infty} q^k \right) = p \frac{d}{dq} \left( \frac{1}{1-q} - 1 \right) = \frac{p}{(1-q)^2} = \frac{1}{p}. \]

**Problem 11.** *(Additional Problem 3).* Find density of \( \frac{U_1}{U_2} \) where \( U_1, U_2 \sim \text{i.i.d. Unif}[0,1] \).

_Solution._ Let

\[ V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} U_1/U_2 \\ U_2 \end{bmatrix}, \]

then conversely,

\[ U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \]

Note that

\[ \frac{\partial u(v)}{\partial v} = \begin{bmatrix} v_2 & v_1 \\ 0 & 1 \end{bmatrix}, \text{ and } \left\| \frac{\partial u(v)}{\partial v} \right\| = |v_2|. \]

Therefore,

\[ f_V(v) = f_U(v_1v_2, v_2) \left\| \frac{\partial u(v)}{\partial v} \right\| \]

\[ = v_2 \mathbb{1}_{[0,1] \times [0,1]}(v_1v_2, v_2) \]

\[ = \begin{cases} v_2 & \text{if } 0 \leq v_2 \leq \min \left\{ \frac{1}{v_1}, 1 \right\}, \ v_1 \geq 0 \\ 0 & \text{o.w.} \end{cases}. \]

Therefore, the density of \( \frac{U_1}{U_2} \) can be derived by marginalizing \( V_2 \): For \( v_1 > 0 \),

\[ f_{V_1}(v_1) = \int_{\mathbb{R}} f_V(v) = \int_{\mathbb{R}} \min \left\{ \frac{1}{v_1}, 1 \right\} v_2 \, dv_2 = \frac{1}{2} \min \left\{ \frac{1}{v_1^2}, 1 \right\}. \]

**Problem 12.** *(Additional Problem 4).* Prove that \( \text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)]. \)
Solution. Note that $E[E[X|Y]] = E[X]$. Therefore, we have

$$\text{Var}(E[X|Y]) = E[E[X|Y]^2] - E[E[X|Y]]^2$$
$$= E[E[X|Y]^2] - E[X]^2,$$

and

$$E[\text{Var}(X|Y)] = E[E[X^2|Y] - E[X|Y]^2]$$
$$= E[X^2] - E[E[X|Y]^2].$$

Hence, summing up two expressions, we would get

$$\text{Var}(E[X|Y]) + E[\text{Var}(X|Y)] = E[X^2] - E[X]^2 = \text{Var}(X).$$