# MATH 281A: Homework #4

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**Problem 1.** *(Problem 2.7.1.22).*

*Solution of* (a). $X \sim \text{Bern}(p)$, where $\frac{1}{4} < p < \frac{3}{4}$.

$$\delta^*(X) = \begin{cases} Y_0 \sim \text{Unif}\left(-\frac{1}{2}, \frac{1}{2}\right) & \text{if } X = 0 \\ Y_1 \sim \text{Unif}\left(\frac{1}{2}, \frac{3}{2}\right) & \text{if } X = 1. \end{cases}$$

Hence,

$$\mathsf{E}[\delta^*(X)] = \mathsf{E}[\delta^*(X)|X=0]\,\mathsf{P}(X=0) + \mathsf{E}[\delta^*(X)|X=1]\,\mathsf{P}(X=1) = \mathsf{E}[Y_0]q + \mathsf{E}[Y_1]p = p,$$

so $\delta^*(X)$ is unbiased for $p$. $\square$

*Solution of* (b). The loss function is given as $L(p,d) = \mathbb{1}\{|d-p| \geq \frac{1}{4}\}$. The risk function for $\delta^*(X)$ is,

$$\begin{aligned}
R_{\delta^*(X)}(p) &= \mathsf{E}_p[L(p, \delta^*(X))] \\
&= \mathsf{E}_p[\mathbb{1}\{|\delta^*(X) - p| \geq \frac{1}{4}\}] \\
&= \mathsf{P}\left(|\delta^*(X) - p| \geq \frac{1}{4}\right) \\
&= \mathsf{P}\left(|Y_0 - p| \geq \frac{1}{4}\right) q + \mathsf{P}\left(|Y_1 - p| \geq \frac{1}{4}\right) p \\
&= q\left(\cancel{\mathsf{P}\left(Y_0 \geq p + \frac{1}{4}\right)} + \mathsf{P}\left(Y_0 \leq p - \frac{1}{4}\right)\right) + p\left(\mathsf{P}\left(Y_1 \geq p + \frac{1}{4}\right) + \cancel{\mathsf{P}\left(Y_1 \leq p - \frac{1}{4}\right)}\right) \\
&= q\left(\left(p - \frac{1}{4}\right) - \left(-\frac{1}{2}\right)\right) + p\left(\frac{3}{2} - \left(p + \frac{1}{4}\right)\right) \\
&= \frac{1}{4} + 2pq = 2p(1-p) + \frac{1}{4}.
\end{aligned}$$

On the other hand,

$$R_X(p) = \mathsf{E}_p[L(p, X)] = \mathsf{E}_p[\mathbb{1}\{|X - p| \geq \frac{1}{4}\}]$$

$$= \mathsf{P}\left(|X-p| \geq \frac{1}{4}\right) = \mathsf{P}\left(X \geq p + \frac{1}{4}\right) + \mathsf{P}\left(X \leq p - \frac{1}{4}\right) = p + q = 1.$$

(Alternatively, since $X$ can only take values of 0 or 1, with $\frac{1}{4} < p < \frac{3}{4}$, $|X-p|$ is greater than $\frac{1}{4}$ with probability 1, thus the risk of estimating $p$ by $X$ equals 1.)
One can easily check that $R_{\delta^*}(p) > R_X(p)$ for all $p \in (\frac{1}{4}, \frac{3}{4})$. □

**Problem 2.** *(Problem 2.7.2.15).*

*Solution.* Noting that

$$\begin{aligned} S_{XY} &= \sum_i (X_i - \overline{X})(Y_i - \overline{Y}) \\ &= \sum_i X_i Y_i - n\overline{XY}, \end{aligned}$$

we have

$$\begin{aligned} \mathsf{E}[S_{XY}] &= n\,\mathsf{E}[X_1Y_1] - \frac{1}{n}\sum_i\sum_j \mathsf{E}[X_iY_j] \\ &= n\,\mathsf{E}[X_1Y_1] - \frac{1}{n}\left(n\,\mathsf{E}[X_1Y_1] + n(n-1)\,\mathsf{E}[X_1]\,\mathsf{E}[Y_1]\right) \\ &= (n-1)\,\mathsf{E}[X_1Y_1] - (n-1)\,\mathsf{E}[X_1]\,\mathsf{E}[Y_1] \\ &= (n-1)\,\mathrm{Cov}(X_1, Y_1). \end{aligned}$$

Hence, $\frac{S_{XY}}{n-1}$ is an unbiased estimator for $\mathrm{Cov}(X, Y)$. □

**Problem 3.** *(Problem 2.7.5.3).*

Table 5.1. $I[\tau(\theta)]$ *for Some Exponential Families*

| Distribution | Parameter $\tau(\theta)$ | $I(\tau(\theta))$ |
|---|---|---|
| $N(\xi, \sigma^2)$ | $\xi$ | $1/\sigma^2$ |
| $N(\xi, \sigma^2)$ | $\sigma^2$ | $1/2\sigma^4$ |
| $b(p, n)$ | $p$ | $n/pq$ |
| $P(\lambda)$ | $\lambda$ | $1/\lambda$ |
| $\Gamma(\alpha, \beta)$ | $\beta$ | $\alpha/\beta^2$ |

*Solution of* (a). $X \sim N(\xi, \sigma^2)$, $i(\xi) = \frac{1}{\sigma^2}$. Since

$$\log f_{\xi,\sigma^2}(x) = C - \frac{1}{2\sigma^2}(x-\xi)^2 - \frac{1}{2}\log\sigma^2,$$

we have

$$\frac{\partial^2}{\partial\xi^2}\log f_{\xi,\sigma^2}(x) = -\frac{1}{\sigma^2}.$$

Hence,

$$i(\xi) = -\,\mathsf{E}\left[\frac{\partial^2}{\partial \xi^2} \log f_{\xi,\sigma^2}(X)\right] = \frac{1}{\sigma^2}. \qquad \square$$

*Solution of* (b). As we derived in the previous problem,

$$\frac{\partial^2}{\partial(\sigma^2)^2} \log f_{\xi,\sigma^2}(x) = -\frac{1}{\sigma^6}(x-\xi)^2 + \frac{1}{2\sigma^4}.$$

Hence,

$$i(\sigma^2) = \frac{1}{\sigma^6}\,\mathsf{E}[(X-\xi)^2] - \frac{1}{2\sigma^4} = \frac{1}{2\sigma^4}. \qquad \square$$

*Solution of* (c). $X \sim \text{Binom}(n,p)$. Then $p_p(k) = \binom{n}{k} p^k (1-p)^{n-k}$, and $\log p_p(k) = C + k\log p + (n-k)\log(1-p)$. Hence,

$$\frac{\partial}{\partial p} \log p_p(k) = \frac{k}{p} - \frac{n-k}{1-p}, \text{ and } \frac{\partial^2}{\partial p^2} \log p_p(k) = -\frac{k}{p^2} - \frac{n-k}{(1-p)^2}.$$

Therefore,

$$i(p) = \frac{1}{p^2}\,\mathsf{E}[X] + \frac{1}{(1-p)^2}\,\mathsf{E}[n-X] = \frac{np}{p^2} + \frac{n(1-p)}{(1-p)^2} = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}. \qquad \square$$

*Solution of* (d). $X \sim \text{Poisson}(\lambda)$. Then $p_\lambda(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, and $\log p_\lambda(x) = -\lambda + x\log\lambda - \log x!$. Hence,

$$\frac{\partial^2}{\partial \lambda^2} \log p_\lambda(x) = \frac{x}{\lambda^2},$$

and thus

$$i(\lambda) = \frac{1}{\lambda^2}\,\mathsf{E}_\lambda[X] = \frac{1}{\lambda}. \qquad \square$$

*Solution of* (e). $X \sim \Gamma(\alpha,\beta)$. Then $f_{\alpha,\beta}(x) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}$ for $x \geq 0$, and $\log f_{\alpha,\beta}(x) = -\alpha\log\beta - \log\Gamma(\alpha) - \frac{x}{\beta} + (\alpha-1)\log x$. Hence,

$$\frac{\partial^2}{\partial \lambda^2} \log f_{\alpha,\beta}(x) = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}.$$

Note that

$$\mathsf{E}_{\alpha,\beta}[X] = \alpha\beta \int_0^\infty \frac{\beta^{-(\alpha+1)\Gamma(\alpha+1)}}{e}^{-x/\beta} x^{(\alpha+1)-1} dx = \alpha\beta.$$

Therefore,

$$i(\beta) = \frac{2}{\beta^3}\,\mathsf{E}_{\alpha,\beta}[X] - \frac{\alpha}{\beta^2} = \frac{2}{\beta^3}\alpha\beta - \frac{\alpha}{\beta^2} = \frac{\alpha}{\beta^2}. \qquad \square$$

**Problem 4.** *(Problem 2.7.5.4).*

*Solution.* (One might think that $i(\sigma) = i(\sigma^2)$, however it is wrong! )
For $\mathcal{N}(0, \sigma^2)$, $i(\sigma^2) = \frac{1}{2\sigma^4}$. By equation (5.11) $I^*(\xi) = I[h(\xi)] \cdot [h'(\xi)]^2$ in chapter 2, with $h(\cdot)$ being we have $i(\sigma) = i(\sigma^2) \cdot (2\sigma)^2 = \frac{2}{\sigma^2}$.
(Understand and memorize equation (5.11) and it will make your life easier during 281 exams.) □

**Problem 5.** *(Problem 2.7.5.5).*

*Solution.* $X \sim \text{Geom}(p)$. Then $p_p(k) = pq^{k-1}$ for $k = 1, 2, \ldots$, where $q = 1 - p$, and $\log p_p(k) = \log p + (k-1)\log(1-p)$. Hence,

$$\frac{\partial^2}{\partial p^2} \log p_p(k) = -\frac{1}{p^2} - \frac{k-1}{(1-p)^2}.$$

Therefore, the Fisher information of $p$ is

$$i(p) = \frac{1}{p^2} + \frac{\mathsf{E}[X] - 1}{(1-p)^2} = \frac{1}{p^2} + \frac{1/p - 1}{(1-p)^2} = \frac{1}{p^2(1-p)}.$$ □

**Problem 6.** *(Problem 2.7.5.6).*

*Solution.* If $X \sim \text{Poisson}(\lambda)$, we know that $\log p_\lambda(x) = -\lambda + x \log \lambda - \log x!$. Then

$$\frac{\partial}{\partial \sqrt{\lambda}} \log p_\lambda(x) = -2\sqrt{\lambda} + \frac{2x}{\sqrt{\lambda}}, \text{ and } \frac{\partial^2}{\partial (\sqrt{\lambda})^2} \log p_\lambda(x) = -2 - \frac{2x}{\lambda}.$$

Hence,

$$i(\sqrt{\lambda}) = 2 + \frac{2}{\lambda}\,\mathsf{E}[X] = 4.$$ □

**Problem 7.** *(Problem 2.7.5.8(b)).*

*Solution.* We want to find a function $g(p)$ for which the amount of information is independent of $p$ for the binomial distribution $\text{Binom}(n, p)$. We know that $\log p_p(x) = C + x \log p + (n-x)\log(1-p)$, and

$$\frac{\partial}{\partial p} \log p_p(x) = \frac{x}{p} - \frac{n-x}{1-p} = \frac{X - np}{p(1-p)}.$$

Furthermore, by chain rule, we have

$$\left(\frac{\partial}{\partial g(p)} \log p_p(X)\right)^2 = \left(\frac{1}{g'(p)} \frac{\partial}{\partial p} \log p_p(X)\right)^2 = \frac{(X - np)^2}{g'(p)^2 p^2 (1-p)^2}.$$

Hence,

$$i(g(p)) = \mathsf{E}\left[\left(\frac{\partial}{\partial g(p)} \log p_p(X)\right)^2\right] = \frac{\text{Var}(X)}{g'(p)^2 p^2 (1-p)^2} = \frac{n}{g'(p)^2 p(1-p)},$$

and if $g'(p)\sqrt{p(1-p)}$ is independent of $p$, then so is $i(g(p))$. After some following algebra,

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2dx}{\sqrt{(1-(1-2x))(1+(1-2x))}} \quad (\cos\theta := 1 - 2x)$$

$$
\begin{aligned}
&= \theta + C \\
&= 2\arcsin(\sqrt{x}) + C,
\end{aligned}
$$

you can verify that a function of the form

$$g(p) = 2a\arcsin(\sqrt{p}) + b$$

for any constants $a$ and $b$ makes $i(g(p))$ independent of $p$. $\square$

**Problem 8.** *(Problem 2.7.5.16).*

*Solution of* (a). Assume that we are given a scale family

$$\mathcal{F} = \left\{ \frac{1}{|\theta|} f\left(\frac{x}{\theta}\right) : \theta > 0 \right\},$$

where $f$ is a density function. We want to verify the following formula for $i(\theta)$, where $\theta$ is a scale parameter.

$$i(\theta) = \frac{1}{\theta^2} \int \left( \frac{y f'(y)}{f(y)} + 1 \right)^2 f(y) dy.$$

Let $f_\theta(x) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right)$. Then, $\log f_\theta(x) = \log f\left(\frac{x}{\theta}\right) - \log\theta$. Hence,

$$\frac{\partial}{\partial\theta} \log f_\theta(x) = \frac{1}{f\left(\frac{x}{\theta}\right)} \frac{\partial}{\partial\theta} f\left(\frac{x}{\theta}\right) - \frac{1}{\theta} = \frac{1}{f\left(\frac{x}{\theta}\right)} \left(-\frac{x}{\theta^2}\right) f'\left(\frac{x}{\theta}\right) - \frac{1}{\theta} = -\frac{1}{\theta}\left( \frac{x}{\theta} \frac{f'\left(\frac{x}{\theta}\right)}{f\left(\frac{x}{\theta}\right)} + 1 \right).$$

Therefore,

$$
\begin{aligned}
i(\theta) = \mathsf{E}\left[ \left( \frac{\partial}{\partial\theta} \log f_\theta(X) \right)^2 \right] &= \frac{1}{\theta^2} \int \left( \frac{x}{\theta} \frac{f'\left(\frac{x}{\theta}\right)}{f\left(\frac{x}{\theta}\right)} + 1 \right)^2 \frac{1}{\theta} f\left(\frac{x}{\theta}\right) dx \\
&= \frac{1}{\theta^2} \int \left( \frac{y f'(y)}{f(y)} + 1 \right)^2 f(y) dy.
\end{aligned}
$$

$\square$

*Solution of* (b). Let $\xi = \log\theta$. Then the information of $\xi$ is

$$i(\xi) = \mathsf{E}\left[ \left( \frac{\partial}{\partial\xi} \log f_\theta(X) \right)^2 \right] = \mathsf{E}\left[ \left( \frac{1}{\frac{\partial\xi}{\partial\theta}} \frac{\partial}{\partial\theta} \log f_\theta(X) \right)^2 \right] = \mathsf{E}\left[ \theta^2 \left( \frac{\partial}{\partial\theta} \log f_\theta(X) \right)^2 \right] = \theta^2 i(\theta),$$

and from the formula we derived above $\theta^2 i(\theta)$ is independent of $\theta$.
(Alternatively, you could make use of equation (5.11). You will end up with the same result.) $\square$

*Solution of* (c). If $X \sim C(0,\theta)$, then $f_\theta(x) = \frac{\theta}{\pi}\frac{1}{\theta^2+x^2}$, and $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. Since $f'(x) = -\frac{2x}{\pi(1+x^2)^2}$, it follows that $x\frac{f'(x)}{f(x)} + 1 = \frac{1-x^2}{1+x^2}$, and the formula gives us

$$i(\theta) = \frac{1}{\theta^2} \int_{-\infty}^{\infty} \left( \frac{1-x^2}{1+x^2} \right)^2 \frac{dx}{\pi(1+x^2)} = \frac{1}{\pi\theta^2} \int_{-\infty}^{\infty} \frac{(1-x^2)^2}{(1+x^2)^3} dx = \frac{1}{2\theta^2}.$$

Here we use the following integration:

$$\int_{-\infty}^{\infty} \frac{(1-x^2)^2}{(1+x^2)^3} dx = \int_{-\pi/2}^{\pi/2} \frac{(1-\tan^2\theta)^2}{\sec^6\theta} \sec^2\theta d\theta$$
$$= \pi - \int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta = \frac{\pi}{2}. \qquad \square$$

**Problem 9.** *(Additional Problem 1).*

*Solution.* Refer to Problem 2.7.5.16 (a). $\square$

**Problem 10.** *(Additional Problem 2).* Prove that if $X \sim \text{geom}(p)$, then $\mathsf{E}[X] = \frac{1}{p}$.

*Solution.* For $k = 1, 2, \ldots$, $\mathsf{P}(X = k) = pq^{k-1}$. Hence,

$$\mathsf{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = p\frac{d}{dq}\left(\sum_{k=1}^{\infty} q^k\right) = p\frac{d}{dq}\left(\frac{1}{1-q} - 1\right) = \frac{p}{(1-q)^2} = \frac{1}{p}. \qquad \square$$

**Problem 11.** *(Additional Problem 3).* Find density of $\frac{U_1}{U_2}$ where $U_1, U_2 \sim$ i.i.d. Unif$[0,1]$.

*Solution.* Let

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} U_1/U_2 \\ U_2 \end{bmatrix},$$

then conversely,

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} V_1V_2 \\ V_2 \end{bmatrix}.$$

Note that

$$\frac{\partial u(v)}{\partial v} = \begin{bmatrix} v_2 & v_1 \\ 0 & 1 \end{bmatrix}, \text{ and } \left\| \frac{\partial u(v)}{\partial v} \right\| = |v_2|.$$

Therefore,

$$\begin{aligned} f_V(v) &= f_U(v_1v_2, v_2) \left\| \frac{\partial u(v)}{\partial v} \right\| \\ &= v_2 \mathbb{1}_{[0,1]\times[0,1]}(v_1v_2, v_2) \\ &= \begin{cases} v_2 & \text{if } 0 \le v_2 \le \min\left\{\frac{1}{v_1}, 1\right\}, v_1 \ge 0 \\ 0 & \text{o.w.} \end{cases}. \end{aligned}$$

Therefore, the density of $V_1\frac{U_1}{U_2}$ can be derived by marginalizing $V_2$: For $v_1 > 0$,

$$f_{V_1}(v_1) = \int_{\mathbb{R}} f_V(v) = \int_0^{\min\left\{\frac{1}{v_1}, 1\right\}} v_2 dv_2 = \frac{1}{2}\min\left\{\frac{1}{v_1^2}, 1\right\}. \qquad \square$$

**Problem 12.** *(Additional Problem 4).* Prove that $\text{Var}(X) = \text{Var}(\mathsf{E}[X|Y]) + \mathsf{E}[\text{Var}(X|Y)]$.

*Solution*. Note that $\mathsf{E}[\mathsf{E}[X|Y]] = \mathsf{E}[X]$. Therefore, we have

$$\begin{aligned}\mathrm{Var}(E[X|Y]) &= \mathsf{E}[\mathsf{E}[X|Y]^2] - \mathsf{E}[\mathsf{E}[X|Y]]^2 \\ &= \mathsf{E}[\mathsf{E}[X|Y]^2] - \mathsf{E}[X]^2,\end{aligned}$$

and

$$\begin{aligned}\mathsf{E}[\mathrm{Var}(X|Y)] &= \mathsf{E}[\mathsf{E}[X^2|Y] - \mathsf{E}[X|Y]^2] \\ &= \mathsf{E}[X^2] - \mathsf{E}[\mathsf{E}[X|Y]^2].\end{aligned}$$

Hence, summing up two expressions, we would get

$$\mathrm{Var}(E[X|Y]) + \mathsf{E}[\mathrm{Var}(X|Y)] = \mathsf{E}[X^2] - \mathsf{E}[X]^2 = \mathrm{Var}(X). \qquad \square$$