Problem 1. 9.6.7 Let \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) be independently distributed according to \( N(\xi, \sigma^2) \) and \( N\sigma(\eta, \tau^2) \), respectively. Find the minimal sufficient statistics for these cases:
(a) \( \xi, \eta, \sigma, \tau \) are arbitrary.
(b) \( \sigma = \tau \) and \( \xi, \eta, \sigma \) are arbitrary.
(c) \( \xi = \eta \) and \( \xi, \sigma, \tau \) are arbitrary.

Solution:
Writing the joint density of \( X_i \)'s and \( Y_j \)'s into the form of exponential family and by Corollary 6.16, it's easy to observe that the minimal sufficient statistics are:
(a) \((\Sigma X_i, \Sigma X_i^2, \Sigma Y_i, \Sigma Y_i^2)\)
(b) \((\Sigma X_i, \Sigma Y_i, \Sigma X_i^2 + \Sigma Y_i^2)\)
(c) \((\Sigma X_i, \Sigma X_i^2, \Sigma Y_i, \Sigma Y_i^2)\)

Problem 1. 9.6.20 (a) Show that in the \( N(\theta, \theta) \) curved exponential family, the sufficient statistic \( T=(\Sigma X_i, \Sigma X_i^2) \) is not minimal.

Solution:
\[
f_\theta(x) = \frac{1}{(\sqrt{2\pi}\theta)^n} \exp(-\sum x_i^2 - \frac{1}{\theta} \sum x_i^2 - \frac{n\theta}{2})
\]
By Factorization criterion, we know that \( \Sigma x_i^2 \) is sufficient, but there does not exist a function \( H \) such that \((\Sigma x_i, \Sigma x_i^2) = T = H(\Sigma x_i^2)\). Therefore, \((\Sigma x_i, \Sigma x_i^2)\) is not minimal.

19.6.20 (b)

Solution:
Similar to the solution of Example 6.17 on page 39, choose the natural parameter \( \eta = (4\theta^3, -6\theta^2, 4\theta) \), choose four points in \( \eta \) such that the resulting \( 3 \times 3 \) difference matrix has rank 3 and is invertible. By Corollary 6.16(ii), \( T \) is minimal sufficient. Note that \( T \) is not complete since it is not of full rank.

Problem 1. 9.6.21 For the situation of Example 6.25(ii), find an unbiased estimator of \( \xi \) based on \( \Sigma X_i \), and another based on \( \Sigma X_i^2 \), hence, deduce that \( T=(\Sigma X_i, \Sigma X_i^2) \) is not complete.

Solution:
Since \( \bar{X} \) and \( \frac{\Gamma((n-1)/2)}{\sqrt{2\Gamma(n/2)}} \sqrt{\frac{S^2}{n(n-1)}} \) are both unbiased estimators for \( \xi \), there exists non-constant function \( f \) such that \( E(f(T)) = 0 \), but \( f(T) \) is not 0 a.e., thus \( T \) is not complete.

Problem 2. 7.2.5 In example 2.1, when both parameters are unknown, show that the UMVU estimator of \( \xi^2 \) is given by \( \delta = \bar{X}^2 - \frac{s^2}{n(n-1)} \) where now \( S^2 = \sum (X_i - \bar{X})^2 \)

Solution:
\((\sum X_i, \sum X_i^2)\) is complete sufficient statistics. \( S^2 = \sum (X_i - \bar{X})^2 = \sum X_i^2 - n\bar{X}^2 \)
\[ E[\delta] = E[X^2] - \frac{E(S^2)}{n(n-1)} \]  
\[ = E[X^2] - \frac{E(\sum X_i^2 - nX^2)}{n(n-1)} \]  
\[ = E[X^2] + \frac{E(X^2)}{n-1} - \frac{E[\sum X_i^2]}{n(n-1)} \]  
\[ = \frac{n}{n-1} E[X^2] - \frac{\sum E[X_i^2]}{n(n-1)} \]  
\[ = \xi^2 \frac{n}{n-1} - \frac{1}{n-1} \xi^2 \]  
\[ = \xi^2 \]  

**Problem 2.** 7.2.15 If \((X_1, Y_1), \ldots, (X_n, X_n)\) are iid according to any bivariate distribution with finite second moments, show that \(S_{xy}/(n-1)\) given by (2.17) is an unbiased estimator of \(\text{cov}(X_i, Y_i)\)

**Solution:**  
\[ E\left[ \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \right] = E\left[ \sum_{i=1}^{n} (X_i Y_i - \bar{X}Y_i - X_i \bar{Y} + \bar{X}Y) \right] \]  
\[ = nE(XY) - nE(\bar{X}Y) \]  
\[ = nE(XY) - nEXEY - n(EXY - E\bar{X}EY) \]  
\[ \text{Cov}(\bar{X}, \bar{Y}) = \frac{1}{n^2} \sum_i \sum_j \text{cov}(X_i, Y_j) \]  

since \(\text{cov}(X_i, Y_j) = 0\) if \(i \neq j\), hence \(\text{Cov}(\bar{X}, \bar{Y}) = \frac{\text{Cov}(X,Y)}{n} \)  
then  
\[ \frac{1}{n-1} E\left[ \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \right] = \frac{1}{n-1} [n\text{Cov}(X, Y) - n\text{Cov}(\bar{X}, \bar{Y})] \]  
\[ = \text{Cov}(X, Y) \]  
thus \(\frac{S_{xy}}{n-1}\) is an unbiased estimator of \(\text{Cov}(X_i, Y_i)\)

**Problem 2.** 7.2.25 Let \(X_1, \ldots, X_m\) and \(Y_1, \ldots, Y_m\) be iid as \(U(0, \theta)\) and \(U(0, \theta')\), respectively. If \(n > 1\), determine the UMVUE of \(\theta/\theta'\).

**Solution:** \((X_{(m)}, Y_{(n)})\) is complete sufficient statistics. First compute the density of \(X_{(m)}\) and \(Y_{(n)}\), then compute \(E(X_{(m)})\) and \(E(Y_{(n)})\), we have \(E(X_{(m)}) = \frac{m}{m+1} \theta\) and \(E(Y_{(n)}) = \frac{n}{n-1} \frac{1}{\theta'}\). Because \(X_{(m)}\) and \(Y_{(n)}\) are independent,  
\[ E\left( \frac{X_{(m)}}{Y_{(n)}} \right) = \frac{mn}{(m+1)(n-1) \frac{1}{\theta'}} \frac{\theta}{\theta'} \]  

From here, you should know what is the UMVUE for \(\frac{\theta}{\theta'}\).
Problem 2. 7.2.27 In example 2.6(b), show that
(a) The bias of the ML estimator is 0 when \( \xi = u \).
(b) At \( \xi = u \), the ML estimator has smaller expected squared error than the UMVU estimator.

Solution:
(a) \( \xi = u \) is equivalent to \( P = \Phi(\xi - u) = 0.5 \), then \( E(\Phi(u - \bar{X})) = E(\Phi(u - \bar{X})) = E(P(Z \leq \xi - \bar{X})) \), where \( Z \) is standard normal and independent of \( \bar{X} \).

\[
E(P(Z \leq \xi - \bar{X})) = E(P(Z + \bar{X} \leq \xi)) = \Phi(0) \text{ since } Z + \bar{X} \text{ follows distribution } N(\xi, 1 + \sigma^2/n)
\]

(b) \( MSE = bias^2 + variance \), since the bias of the ML estimator and UMVUE are both 0 and \( u - \bar{X} \) is always closer to 0 than \( \sqrt{n-1}(u - \bar{X}) \), and \( \Phi(X) \) is monotonic increasing, the ML estimator has smaller variance and thus smaller expected squared error than the UMVUE.

Problem 2. 7.3.18 If \( X \) has the Poisson distribution \( P(\theta) \), show that \( 1/\theta \) does not have an unbiased estimator.

Solution:
Suppose \( 1/\theta \) has an unbiased estimator \( \delta(x) \), then \( E(\delta(x)) = \sum_{x=0}^{\infty} \delta(x) \frac{e^{-\theta} \theta^x}{x!} = \theta^{-1} \)

\[
\theta^{-1} \sum_{y=0}^{\infty} \frac{\theta^y}{y!} = \sum_{x=0}^{\infty} \delta(x) \frac{\theta^x}{x!}
\]

\[
\sum_{y=0}^{\infty} \frac{\theta^y}{y!} = \sum_{x=0}^{\infty} \delta(x) \frac{\theta^x}{x!}
\]

\[
\sum_{x=1}^{\infty} \frac{\theta^x}{(x+1)!} = \sum_{x=0}^{\infty} \delta(x) \frac{\theta^x}{x!}
\]

\[
\frac{1}{\theta} + \sum_{x=0}^{\infty} \frac{\theta^x}{x!} \frac{1}{x+1} = \sum_{x=0}^{\infty} \delta(x) \frac{\theta^x}{x!}
\]

obviously, whatever \( \delta(x) \) is, the equality cannot be satisfied, hence \( 1/\theta \) does not have an unbiased estimator.