Problem 1. (Problem 1.9.6.3).
Solution. Let \( p_\theta(x) = c(\theta)f(x)\mathbb{1}_{(0,\theta)}(x) \), and \( X_1, \ldots, X_n \sim \text{i.i.d.} \ p_\theta \). Then the joint density is

\[
f_\theta(x_1, \ldots, x_n) = c(\theta)^n \prod_{i=1}^n f(x_i) \mathbb{1}_{(0,\theta)}(x_i)
\]

\[
= c(\theta)^n \mathbb{1}_{(0,\theta)}(x_{(n)}) \prod_{i=1}^n f(x_i)
\]

Hence, by Neyman-Fisher Factorization Theorem, \( X_{(n)} \) is sufficient for \( \theta \). \( \square \)

Problem 2. (Problem 1.9.6.4).
Solution. Similarly,

\[
f_{\xi,\eta}(x_1, \ldots, x_n) = c(\xi,\eta)^n \mathbb{1}_{(\xi,\infty)}(x_{(1)}) \mathbb{1}_{(-\infty,\eta)}(x_{(n)}) \prod_{i=1}^n f(x_i)
\]

Hence, by Neyman-Fisher Factorization Theorem, \( (X_{(1)}, X_{(n)}) \) is sufficient for \( (\xi, \eta) \). \( \square \)

Problem 3. (Problem 1.9.6.5). Example 6.11: Let \( X_1, \ldots, X_n \sim \text{i.i.d.} \ N(0, \sigma^2) \).

\[
T_1(X) = (X_1, \ldots, X_n),
\]

\[
T_2(X) = (X_1^2, \ldots, X_n^2),
\]

\[
T_3(X) = (X_1^2 + \cdots + X_m^2, X_{m+1}^2 + \cdots + X_n^2),
\]

\[
T_4(X) = X_1^2 + \cdots + X_n^2.
\]

Solution. Note that \( T_4(X) \) can be expressed as a function of \( T_1(X), T_2(X), T_3(X) \) respectively. Therefore, from the definition of sufficiency, we only need to show that \( T_4(X) \) is sufficient for \( \sigma^2 \). However, since

\[
f_{\sigma^2}(x_1, \ldots, x_n) = (\sqrt{2\pi\sigma^2})^{-n} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \right),
\]

\( T_4(X) \) is sufficient. \( \square \)
**Problem 4.** *(Problem 1.9.7.5).*

Solution of (a). If $\phi$ is convex on $(a,b)$, then so is $e^{\phi(x)}$ because for any $\lambda \in [0,1]$ and $x,y \in (a,b)$ using the convexity of $x \mapsto e^x$ and Jensen’s Inequality, the following holds

$$\lambda e^{\phi(x)} + (1-\lambda)e^{\phi(y)} \geq e^{\lambda \phi(x) + (1-\lambda) \phi(y)} \geq e^{\phi(\lambda x + (1-\lambda)y)}.$$

Solution of (b). If we let $\phi(x) = x$ on $x \in (0,\infty)$, which is convex, then, however, $\log \phi(x) = \log x$ is concave.

**Problem 5.** *(Problem 1.9.7.10).* Let $U \sim \text{Unif}(0,1)$, and let $F$ be a distribution function on $\mathbb{R}$.

Solution of (a). Assuming $F$ is continuous and strictly increasing, then $F^{-1}$ exists. Let $V = F^{-1}(U)$. Then

$$F_V(v) = \mathbb{P}(F^{-1}(Y) \leq v) = \mathbb{P}(U \leq F(v)) = F(v),$$

so $F^{-1}(U)$ has distribution function $F$.

Solution of (b). (Reference: pp.34-35, Probability with Martingales by David Williams(1991)) Now let $F$ be an arbitrary distribution function. Then $F$ is right continuous and nondecreasing. Since, in general, $F$ might not have an inverse function, so define $F^{-1}$ as follows:

$$F^{-1}(\omega) = \sup \{ z : F(z) \leq \omega \}.$$

We claim that

$$F^{-1}(\omega) \leq x \iff F(x) \geq \omega.$$

First, by definition of $F^{-1}$, we have

$$F(x) \geq \omega \implies F^{-1}(\omega) \leq x.$$

Now, on the other hand, by right continuity of $F$, we have $F(F^{-1}(\omega)) \geq \omega$. Thus,

$$F^{-1}(\omega) \leq x \implies \omega \leq F(F^{-1}(\omega)) \leq F(x) \implies \omega \leq F(x).$$

So, the claim is true, and henceforth $V := F^{-1}(U)$ has distribution function $F$, because

$$F_V(v) = \mathbb{P}(V \leq v) = \mathbb{P}(F^{-1}(U) \leq v) = \mathbb{P}(U \leq F(v)) = F(v).$$

**Problem 6.** *(Problem 1.9.5.1).*

Solution.

$$f(x|\eta) = e^{-A(\eta)}e^{\eta x}h(x).$$

We want $e^{\eta x}h(x)$ to be integrable for each case.
Thus, it is integrable if and only if $|\eta| < 1$, and thus this is the natural parameter space.

\[ \int_{-\infty}^{\infty} e^{\eta x - |x|} dx = \int_{0}^{\infty} e^{(\eta - 1)x} dx + \int_{-\infty}^{0} e^{(\eta + 1)x} dx. \]

It is easy to check that it is integrable if and only if $|\eta| \leq 1$, since $1/(1 + x^2)$ is integrable.

**Problem 7.** (Problem 1.9.5.2).

Solution. We have $f(x)(\eta_1, \eta_2) = \exp((\eta_1 + \eta_2)T(x) - A(\eta))h(x) = \exp((\eta_1 + c + \eta_2 - c)T(x) - A(\eta))h(x) = f(x)(\eta_1 + c, \eta_2 - c)$ for any $c$. Therefore, from Definition 5.2 of unidentifiability, $(\eta_1, \eta_2)$ is unidentifiable, and thus unestimable.

**Problem 8.** (Problem 1.9.5.12).

Solution. $P(X = x) = \frac{a(x)\theta^x}{C(\theta)}$ for $x = 0, 1, \ldots; a(x) \geq 0; \theta > 0$. In the form of exponential family,

\[ P(X = x) = \exp((\log \theta)x - \log C(\theta))a(x). \]

So, a power series distribution is an exponential family with $s = 1, \eta = \log \theta$ and $T = X$. Note that $C(\theta) = \sum_{x=0}^{\infty} a(x)\theta^x$. The moment generating function of $X$ is

\[ M_X(u) = E[e^{ux}] = \sum_{x=0}^{\infty} e^{ux} \frac{a(x)\theta^x}{C(\theta)} = \frac{1}{C(\theta)} \sum_{x=0}^{\infty} a(x)(\theta e^u)^x = \frac{C(\theta e^u)}{C(\theta)}. \]

**Problem 9.** (Problem 1.9.5.13).

Solution of (a). Binomial:

\[ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \mathbb{1}_{[0,n]}(x) \]

\[ = \binom{n}{x} (1 - p)^n \left( \frac{p}{1 - p} \right)^x \mathbb{1}_{[0,n]}(x). \]

Thus, $\theta = \frac{p}{1 - p}$ and $C(\theta) = (1 - p)^{-n} = (1 - \theta)^n$.

Solution of (b). Geometric: for $x = 0, 1, 2, \ldots$

\[ P(X = x) = pq^x. \]

Thus, $\theta = q = 1 - p$ and $C(\theta) = 1/p = 1/(1 - \theta)$.
Solution of (c). Poisson:

\[ P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!} \]

Thus, \( \theta = \lambda \) and \( C(\theta) = e^\lambda = \log \theta \).

**PROBLEM 10. (Problem 1.9.5.14).**

Solution. We have \( a(x) = 1/x \) and \( C(\theta) = -\log(1 - \theta) \) where \( 0 < \theta < 1 \) Hence, for \( x = 1, 2, \ldots \)

\[ P(X = x) = -\frac{\theta^x}{x \log(1 - \theta)} . \]

From the formula we derived above,

\[ M_X(u) = \frac{C(\theta e^u)}{C(\theta)} = \frac{\log(1 - \theta e^u)}{\log(1 - \theta)} = E[e^{uX}] . \]

We can calculate moments of \( X \) using this mgf.

\[ \mathbb{E}[X^k] = \frac{d^k}{du^k} M_X(u) \bigg|_{u=0} . \]

Thus,

\[ \mathbb{E}[X] = \frac{d}{du} M_X(u) \bigg|_{u=0} = \frac{-\theta e^u}{1 - \theta e^u \log(1 - \theta)} \bigg|_{u=0} = \frac{-\theta}{(1 - \theta) \log(1 - \theta)} , \]

and

\[ \mathbb{E}[X^2] = \frac{d^2}{du^2} M_X(u) \bigg|_{u=0} = \frac{-\theta e^u - \frac{1}{2(1 - \theta e^u)^2 \log(1 - \theta)}}{1 - \theta e^u \log(1 - \theta)} \bigg|_{u=0} = \frac{-\theta}{2(1 - \theta)^2 \log(1 - \theta)} . \]

Putting things together, we finally get

\[ \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{\theta}{((1 - \theta) \log(1 - \theta))^2} \left( \log \frac{1}{1 - \theta} - \theta \right) . \]

**PROBLEM 11. (Problem 1.9.5.31).**

Solution. \( X_1, \ldots, X_n \sim \text{i.i.d. } \Gamma(\alpha, \beta) \). Then the joint density is given as

\[ f_{\alpha, \beta}(x_1, \ldots, x_n) = \prod_{i=1}^n \beta^{\alpha - 1} \frac{e^{-x_i/\beta} \Gamma(\alpha)}{\Gamma(\alpha) x_i^\alpha} = \beta^{n\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha)^n} \exp \left( -\frac{1}{\beta} \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \log x_i \right) 1_{(0, \infty)}(x_i) . \]

Hence, this is a two-parameter exponential family with \( \eta = (-\frac{1}{\beta}, \alpha - 1) \), \( T = (\sum_{i=1}^n x_i, \sum_{i=1}^n \log X_i) \), \( B(\alpha, \beta) = n(\alpha \log \beta + \log \Gamma(\alpha)) \), and \( h(x_1, \ldots, x_n) = \prod_{i=1}^n 1_{(0, \infty)}(x_i) . \)

**PROBLEM 12. (Problem 1.9.5.32).**
Solution. \( Y \sim \Gamma(\alpha, \beta) \). For \( y > 0 \),

\[
f_Y(y) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} e^{-y/\beta} y^{\alpha-1}.
\]

Let \( X = c \log Y \), then \( Y = e^{X/c} \). Therefore,

\[
f_X(x) = f_Y(e^{x/c}) \frac{de^{x/c}}{dx} = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} e^{-e^{x/c}/\beta} e^{(x-1)/c} \frac{1}{c} e^{x/c} = \frac{\beta^{-\alpha}}{c\Gamma(\alpha)} \exp \left( \frac{\alpha x}{c} - \frac{1}{\beta} e^{x/c} \right).
\]

Thus, it defines an exponential family for fixed \( \alpha \) and varying \( \beta \).

**Problem 13.** (Additional problem 1). Let \( X_1, \ldots, X_n \sim \text{i.i.d. geom}(p) \). Using Rao-Blackwell’s Theorem, find unbiased estimators for the following target functions.

(a) \( g_1(p) = \frac{1}{p} \).

(b) \( g_2(p) = p \).

Solution of (a). We know that \( T = \sum X_i \) is sufficient for \( p \), since joint pmf can be factorized as

\[
f_p(x_1, \ldots, x_n) = p^n q^{\sum x_i - n}.
\]

Now set our naive guess as \( \delta(X) = \overline{X} \). Then \( E[\delta(X)] = E[\overline{X}] = p \), so it is unbiased. Also, it only depends on \( T \), so \( \delta(X) \) would remain same after Rao-Blackwellizing it.

Solution of (b). Set our first naive guess as \( \delta(X) = 1 \{ X = 1 \} \). Obviously, \( E[\delta(X)] = P(X = 1) = p \), so it is unbiased. Now, using Rao-Blackwell Theorem, we can find a better unbiased estimator \( \eta(T) \).

\[
\eta(t) = E[\delta(X)|T = t] = P \left( X_1 = 1 \left| \sum_{i=1}^{n} X_i = t \right. \right) = \frac{P \left( X_1 = 1, \sum_{i=1}^{n} X_i = t \right)}{P \left( \sum_{i=1}^{n} X_i = t \right)}
\]

\[
= \frac{P \left( X_1 = 1 \right) P \left( \sum_{i=2}^{n} X_i = t - 1 \right)}{P \left( \sum_{i=1}^{n} X_i = t \right)}
\]

\[
= \frac{p \cdot p^{n-1} q^{t-n} \binom{t-2}{n-2}}{p^n q^{t-n} \binom{t-1}{n-1}}
\]

\[
= \frac{t - 1}{n - 1},
\]

thus

\[
\eta(T) = \frac{n - 1}{T - 1} = \frac{n - 1}{\sum_{i=1}^{n} X_i - 1}.
\]
Note that we use the fact that for $t \geq n$
\[ P \left( \sum_{i=1}^{n} X_i = t \right) = p^n q^{t-n} \binom{t-1}{n-1}. \]

This can be proved by induction. This holds trivially when $n = 1$. Suppose this holds for $n - 1$ for $n \geq 2$. Then for $t \geq n$, we have
\[
P(X_1 + \ldots + X_n = t) = \sum_{k=n-1}^{t-1} P(X_1 + \ldots + X_{n-1} = k) P(X_n = t - k)
= p^n q^{t-n} \sum_{k=n-1}^{t-1} \binom{k-1}{n-2}
= p^n q^{t-n} \sum_{k=n-2}^{t-2} \binom{k}{n-2}
= p^n q^{t-n} \binom{t-1}{n-1},
\]
because the following identity holds
\[
\sum_{k=r}^{n} \binom{k}{r} = \sum_{l=0}^{n-r} \binom{l+r}{r} = \binom{n+1}{r+1}.
\]

**Problem 14. (Additional problem 2).** Two toy examples:

(a) Show that $X \sim \begin{cases} 1 \text{ w.p. } \theta \\ 2 \text{ w.p. } 2\theta \\ 3 \text{ w.p. } 1 - 3\theta \end{cases}$ is not complete where $\theta \in [0, \frac{1}{3}]$.

(b) Show that $X \sim \begin{cases} 1 \text{ w.p. } \theta \\ 2 \text{ w.p. } \theta^2 \\ 3 \text{ w.p. } 1 - \theta - \theta^2 \end{cases}$ is complete where $\theta \in [0, \frac{1}{100}]$.

Solution of (a). We can find a non-trivial unbiased estimator for 0, for example, $\delta(X) = 3 \mathbb{1}\{X = 1\} + \mathbb{1}\{X = 3\} - 1$. One can easily check that
\[ E_\theta \delta(X) = 3 P(X = 1) + P(X = 3) - 1 = 0. \]
\[ \text{Solution of (b). Suppose } E_\theta \delta(X) = 0 \text{ for all } \theta \in [0, \frac{1}{100}]. \text{ This implies that for all } \theta \]
\[ \delta(1)\theta + \delta(2)\theta^2 + \delta(3)(1 - \theta - \theta^2) = 0, \]

or
\[ (\delta(2) - \delta(3))\theta^2 + (\delta(1) - \delta(3))\theta + \delta(3) = 0. \]
This implies that \( \delta(2) = \delta(3) = \delta(1) = 0 \) to make the polynomial constantly 0 for a closed interval, so \( X \) is complete. \( \square \)

**Problem 15.** (Additional problem 3). (Uniform problem) Let \( X_1, \ldots, X_n \sim \text{i.i.d. Unif}[\alpha, \beta] \). We want to estimate \( g(\alpha, \beta) = \alpha + \beta^2 \).

(a) Show that \( T = (X_1(X_1), X_n(X_n)) \) is sufficient.

(b) Calculate \( E[X_1|X_1(X_1), X_n(X_n)] \).

(c) Show that the Rao-Blackwellized unbiased estimator for this uniform problem \( E[X|X_1(X_1), X_n(X_n)] \)
is also unbiased for true mean for all symmetric distributions in \( \mathcal{L}^2 \).

\[ \text{Solution of (a). Since a single random variable follows } f_{\alpha,\beta}(x) = \frac{1}{\beta - \alpha} \mathbb{1}_{[\alpha, \beta]}(x), \text{ we have joint distribution of } X_1, \ldots, X_n \]
\[ f_{\alpha,\beta}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\beta - \alpha} \mathbb{1}_{[\alpha, \beta]}(x_i) = \prod_{i=1}^{n} \frac{1}{\beta - \alpha} \mathbb{1}_{[\alpha, \beta]}(x(i)) = \frac{1}{(\beta - \alpha)^n} \mathbb{1}_{[\alpha, \beta]}(x(1)) \mathbb{1}_{[\alpha, \beta]}(x(n)). \]
Therefore, by Neyman-Fisher factorization Theorem, \( T = (X_1(X_1), X_n(X_n)) \) is a sufficient statistic. \( \square \)

\[ \text{Solution of (b). Note that given } T, X_1, \ldots, X_n \text{ are still identically distributed, and } X(i)|[X_1(X_1), X_n(X_n)] \sim \text{Unif}[X_1(X_1), X_n(X_n)] \text{ for } 2 \leq i \leq n - 1. \text{ Hence,} \]
\[ E[X_1|X_1(X_1), X_n(X_n)] = \frac{1}{n} E[X_1 + \ldots + X_n|X_1(X_1), X_n(X_n)] \]
\[ = \frac{1}{n} E[X_1 + \ldots + X_n|X_1(X_1), X_n(X_n)] \]
\[ = \frac{1}{n} \left( X_1(X_1) + X_n(X_n) + (n - 2)\frac{X_1(X_1) + X_n(X_n)}{2} \right) \]
\[ = \frac{X_1(X_1) + X_n(X_n)}{2}. \]

\[ \text{Solution of (c). The resulting Rao-Blackwellized unbiased estimator is} \]
\[ E[X|X_1(X_1), X_n(X_n)] = \frac{X_1(X_1) + X_n(X_n)}{2}. \]
Now consider an arbitrary symmetric distribution with mean \( \theta \). Then this is equivalent to that \( X - \theta \) is symmetric about 0, or equivalently \( X - \theta \sim \theta - X \). Therefore, we have
\[ (X_1 - \theta) + (X_n - \theta) \sim (\theta - X_1) + (\theta - X_n), \]
and this implies that $\mathbb{E}[(X_1 - \theta) + (X_n - \theta)] = 0$. Henceforth,

$$\mathbb{E} \left[ \frac{X_1 + X_n}{2} \right] = \theta,$$

so $\frac{X_1 + X_n}{2}$ is unbiased for $\theta$. \qed