MATH 281A: Homework #5

Jongha Ryu

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Problem 1. *(Problem 1.9.6.3).*
Solution. Let $p_\theta(x) = c(\theta)f(x)\mathbb{1}_{(0,\theta)}(x)$, and $X_1, \ldots, X_n \sim$ i.i.d. p_θ. Then the joint density is

$$\begin{aligned} f_\theta(x_1, \ldots, x_n) &= c(\theta)^n \prod_{i=1}^n f(x_i)\mathbb{1}_{(0,\theta)}(x_i) \\ &= \underbrace{c(\theta)^n \mathbb{1}_{(0,\theta)}(x_{(n)})}_{g_\theta(t(x_1,\ldots,x_n))} \underbrace{\prod_{i=1}^n f(x_i)}_{h(x_1,\ldots,x_n)} . \end{aligned}$$

Hence, by Neyman-Fisher Factorization Theroem, $X_{(n)}$ is sufficient for θ. □

Problem 2. *(Problem 1.9.6.4).*
Solution. Similarly,

$$f_{\xi,\eta}(x_1, \ldots, x_n) = \underbrace{c(\xi,\eta)^n \mathbb{1}_{(\xi,\infty)}(x_{(1)})\mathbb{1}_{(-\infty,\eta)}(x_{(n)})}_{g_{\xi,\eta}(t(x_1,\ldots,x_n))} \underbrace{\prod_{i=1}^n f(x_i)}_{h(x_1,\ldots,x_n)} .$$

Hence, by Neyman-Fisher Factorization Theroem, $(X_{(1)}, X_{(n)})$ is sufficient for (ξ, η). □

Problem 3. *(Problem 1.9.6.5).* Example 6.11: Let $X_1, \ldots, X_n \sim$ i.i.d. $\mathcal{N}(0, \sigma^2)$.

$$\begin{aligned} &T_1(X) = (X_1, \ldots, X_n), \\ &T_2(X) = (X_1^2, \ldots, X_n^2), \\ &T_3(X) = (X_1^2 + \cdots + X_m^2, X_{m+1}^2 + \cdots + X_n^2), \\ &T_4(X) = X_1^2 + \cdots + X_n^2. \end{aligned}$$

Solution. Note that $T_4(X)$ can be expressed as a function of $T_1(X), T_2(X), T_3(X)$ respectively. Therefore, from the definition of sufficiency, we only need to show that $T_4(X)$ is sufficient for σ^2. However, since

$$f_{\sigma^2}(x_1, \ldots, x_n) = (\sqrt{2\pi\sigma^2})^{-n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right),$$

$T_4(X)$ is sufficient. □

Problem 4. *(Problem 1.9.7.5).*

Solution of (a). If ϕ is convex on (a,b), then so is $e^{\phi(x)}$ because for any $\lambda \in [0,1]$ and $x, y \in (a,b)$using the convexity of $x \mapsto e^x$ and Jensen's Inequality, the following holds

$$\lambda e^{\phi(x)} + (1-\lambda)e^{\phi(y)} \geq e^{\lambda\phi(x)+(1-\lambda)\phi(y)} \geq e^{\phi(\lambda x+(1-\lambda)y)}. \qquad \square$$

Solution of (b). If we let $\phi(x) = x$ on $x \in (0,\infty)$, which is convex, then, however, $\log\phi(x) = \log x$ is concave. $\square$

Problem 5. *(Problem 1.9.7.10).*Let $U \sim \mathrm{Unif}(0,1)$, and let F be a distribution function on $\mathbb{R}$.

Solution of (a). Assumming F is continuous and strictly increasing, then F^{-1} exists. Let $V = F^{-1}(U)$. Then

$$F_V(v) = \mathsf{P}(F^{-1}(Y) \leq v) = \mathsf{P}(U \leq F(v)) = F(v),$$

so $F^{-1}(U)$ has distribution function F. $\square$

Solution of (b). (Reference: pp.34-35, Probability with Martingales by David Williams(1991)) Now let F be an arbitrary distribution function. Then F is right continuous and nondecreasing. Since, in general, F might not have an inverse function, so define F^{-1} as follows:

$$F^{-1}(\omega) = \sup\{z \colon F(z) \leq \omega\}.$$

We claim that

$$F^{-1}(\omega) \leq x \Leftrightarrow F(x) \geq \omega.$$

First, by definition of F^{-1}, we have

$$F(x) \geq \omega \implies F^{-1}(\omega) \leq x.$$

Now, on the other hand, by right continuity of F, we have $F(F^{-1}(\omega)) \geq \omega$. Thus,

$$\begin{aligned} F^{-1}(\omega) \leq x &\implies \omega \leq F(F^{-1}(\omega)) \leq F(x) \\ &\implies \omega \leq F(x). \end{aligned}$$

So, the claim is true, and henceforth $V := F^{-1}(U)$ has distribution function F, because

$$F_V(v) = \mathsf{P}(V \leq v) = \mathsf{P}(F^{-1}(U) \leq v) = \mathsf{P}(U \leq F(v)) = F(v). \qquad \square$$

Problem 6. *(Problem 1.9.5.1).*

Solution.

$$f(x|\eta) = e^{-A(\eta)}e^{\eta x}h(x).$$

We want $e^{\eta x}h(x)$ to be integrable for each case.

Solution of (a). $h(x) = e^{-|x|}$.

$$\int_{-\infty}^{\infty} e^{\eta x - |x|} dx = \int_0^{\infty} e^{(\eta-1)x} dx + \int_{-\infty}^{0} e^{(\eta+1)x} dx.$$

Note that it is integrable if and only if $|\eta| < 1$, and thus this is the natural parameter space. □

Solution of (b). $h(x) = \frac{e^{-|x|}}{1+x^2}$.

$$\int_{-\infty}^{\infty} \frac{e^{\eta x - |x|}}{1+x^2} dx = \int_0^{\infty} \frac{e^{(\eta-1)x}}{1+x^2} dx + \int_{-\infty}^{0} \frac{e^{(\eta+1)x}}{1+x^2} dx.$$

It is easy to check that it is integrable if and only if $|\eta| \leq 1$, since $1/(1+x^2)$ is integrable. □

Problem 7. *(Problem 1.9.5.2).*

Solution. We have $f(x|(\eta_1, \eta_2)) = \exp((\eta_1 + \eta_2)T(x) - A(\eta))h(x) = \exp((\eta_1 + c + \eta_2 - c)T(x) - A(\eta))h(x) = f(x|(\eta_1 + c, \eta_2 - c))$ for any c. Therefore, from Definition 5.2 of unidentifiablity, (η_1, η_2) is unidentifiable, and thus unestimable. □

Problem 8. *(Problem 1.9.5.12).*

Solution. $\mathsf{P}(X = x) = \frac{a(x)\theta^x}{C(\theta)}$ for $x = 0, 1, \ldots$; $a(x) \geq 0$; $\theta > 0$. In the form of exponential family,

$$\mathsf{P}(X = x) = \exp((\log\theta)x - \log C(\theta))a(x).$$

So, a power series distribution is an exponential family with $s = 1, \eta = \log\theta$ and $T = X$. Note that $C(\theta) = \sum_{x=0}^{\infty} a(x)\theta^x$. The moment generating function of X is

$$M_X(u) = \mathsf{E}[e^{uX}] = \sum_{x=0}^{\infty} e^{ux} \frac{a(x)\theta^x}{C(\theta)} = \frac{1}{C(\theta)} \sum_{x=0}^{\infty} a(x)(\theta e^u)^x = \frac{C(\theta e^u)}{C(\theta)}.$$ □

Problem 9. *(Problem 1.9.5.13).*

Solution of (a). Binomial:

$$\begin{aligned}\mathsf{P}(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{[0,n]}(x) \\ &= \binom{n}{x} (1-p)^n \left(\frac{p}{1-p}\right)^x \mathbb{1}_{[0,n]}(x).\end{aligned}$$

Thus, $\theta = \frac{p}{1-p}$ and $C(\theta) = (1-p)^{-n} = (1-\theta)^n$. □

Solution of (b). Geometric: for $x = 0, 1, 2, \ldots$

$$\mathsf{P}(X = x) = pq^x.$$

Thus, $\theta = q = 1 - p$ and $C(\theta) = 1/p = 1/(1-\theta)$. □

Solution of (c). Poisson:

$$\mathsf{P}(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

Thus, $\theta = \lambda$ and $C(\theta) = e^{\lambda} = \log\theta$. $\square$

Problem 10. *(Problem 1.9.5.14).*

Solution. We have $a(x) = 1/x$ and $C(\theta) = -\log(1-\theta)$ where $0 < \theta < 1$ Hence, for $x = 1, 2, \ldots$

$$\mathsf{P}(X = x) = -\frac{\theta^x}{x\log(1-\theta)}.$$

From the formula we derived above,

$$M_X(u) = \frac{C(\theta e^u)}{C(\theta)} = \frac{\log(1-\theta e^u)}{\log(1-\theta)} = \mathsf{E}[e^{uX}].$$

We can calculate moments of X using this mgf.

$$\mathsf{E}[X^k] = \frac{d^k}{du^k}M_X(u)\bigg|_{u=0}.$$

Thus,

$$\mathsf{E}[X] = \frac{d}{du}M_X(u)\bigg|_{u=0} = \frac{-\theta e^u}{1-\theta e^u}\frac{1}{\log(1-\theta)}\bigg|_{u=0} = \frac{-\theta}{(1-\theta)\log(1-\theta)},$$

and

$$\mathsf{E}[X^2] = \frac{d^2}{du^2}M_X(u)\bigg|_{u=0} = \frac{-\theta e^u}{2(1-\theta e^u)^2}\frac{1}{\log(1-\theta)}\bigg|_{u=0} = \frac{-\theta}{2(1-\theta)^2\log(1-\theta)}.$$

Putting things together, we finally get

$$\mathrm{Var}(X) = \mathsf{E}[X^2] - \mathsf{E}[X]^2 = \frac{\theta}{((1-\theta)\log(1-\theta))^2}\left(\log\frac{1}{1-\theta} - \theta\right). \qquad \square$$

Problem 11. *(Problem 1.9.5.31).*

Solution. $X_1, \ldots, X_n \sim$ i.i.d. $\Gamma(\alpha, \beta)$. Then the joint density is given as

$$f_{\alpha,\beta}(x_1, \ldots, x_n) = \prod_{i=1}^n \frac{\beta^{-\alpha}}{\Gamma(\alpha)}e^{-x_i/\beta}x_i^{\alpha-1} = \frac{\beta^{-n\alpha}}{\Gamma(\alpha)^n}\exp\left(-\frac{1}{\beta}\sum_{i=1}^n x_i + (\alpha-1)\sum_{i=1}^n \log x_i\right)\mathbb{1}_{(0,\infty)}(x_i).$$

Hence, this is a two-parameter exponential family with $\eta = (-\frac{1}{\beta}, \alpha-1)$, $T = (\sum_{i=1}^n X_i, \sum_{i=1}^n \log X_i)$, $B(\alpha, \beta) = n(\alpha\log\beta + \log\Gamma(\alpha))$, and $h(x_1, \ldots, x_n) = \prod_{i=1}^n \mathbb{1}_{(0,\infty)}(x_i)$. $\square$

Problem 12. *(Problem 1.9.5.32).*

Solution. $Y \sim \Gamma(\alpha, \beta)$. For $y > 0$,

$$f_Y(y) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} e^{-y/\beta} y^{\alpha-1}.$$

Let $X = c \log Y$, then $Y = e^{X/c}$. Therefore,

$$f_X(x) = f_Y(e^{x/c}) \frac{de^{x/c}}{dx} = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} e^{-e^{x/c}/\beta} e^{x(\alpha-1)/c} \frac{1}{c} e^{x/c} = \frac{\beta^{-\alpha}}{c\Gamma(\alpha)} \exp\left(\frac{\alpha x}{c} - \frac{1}{\beta} e^{x/c}\right).$$

Thus, it defines an exponential family for fixed α and varying β. □

Problem 13. *(Additional problem 1).*Let $X_1, \ldots, X_n \sim$ i.i.d. geom(p). Using Rao-Blackwell's Theorem, find unbiased estimators for the following target functions.

(a) $g_1(p) = \frac{1}{p}$.
(b) $g_2(p) = p$.

Solution of (a). We know that $T = \sum_i X_i$ is sufficient for p, since joint pmf can be factorized as

$$f_p(x_1, \ldots, x_n) = p^n q^{\sum_i x_i - n}.$$

Now set our naive guess as $\delta(X) = \overline{X}$. Then $\mathsf{E}[\delta(X)] = \mathsf{E}[\overline{X}] = p$, so it is unbiased. Also, it only depends on T, so $\delta(X)$ would remain same after Rao-Blackwellizing it. □

Solution of (b). Set our first naive guess as $\delta(X) = \mathbb{1}\{X = 1\}$. Obivously, $\mathsf{E}[\delta(X)] = \mathsf{P}(X = 1) = p$, so it is unbiased. Now, using Rao-Blackwell Theorem, we can find a better unbiased estimator $\eta(T)$.

$$\begin{aligned}
\eta(t) &= \mathsf{E}[\delta(X) | T = t] \\
&= \mathsf{P}\left(X_1 = 1 \middle| \sum_{i=1}^n X_i = t\right) \\
&= \frac{\mathsf{P}\left(X_1 = 1, \sum_{i=1}^n X_i = t\right)}{\mathsf{P}\left(\sum_{i=1}^n X_i = t\right)} \\
&= \frac{\mathsf{P}\left(X_1 = 1\right) \mathsf{P}\left(\sum_{i=2}^n X_i = t-1\right)}{\mathsf{P}\left(\sum_{i=1}^n X_i = t\right)} \\
&= \frac{p \cdot p^{n-1} q^{t-n} \binom{t-2}{n-2}}{p^n q^{t-n} \binom{t-1}{n-1}} \\
&= \frac{t-1}{n-1},
\end{aligned}$$

thus

$$\eta(T) = \frac{n-1}{T-1} = \frac{n-1}{\sum_{i=1}^n X_i - 1}.$$

Note that we use the fact that for $t \geq n$

$$\mathsf{P}\left(\sum_{i=1}^{n} X_i = t\right) = p^n q^{t-n} \binom{t-1}{n-1}.$$

This can be proved by induction. This holds trivially when $n = 1$. Suppose this holds for $n-1$ for $n \geq 2$. Then for $t \geq n$, we have

$$\begin{aligned}
\mathsf{P}(X_1 + \ldots + X_n = t) &= \sum_{k=n-1}^{t-1} \mathsf{P}(X_1 + \ldots + X_{n-1} = k)\, \mathsf{P}(X_n = t-k) \\
&= p^n q^{t-n} \sum_{k=n-1}^{t-1} \binom{k-1}{n-2} \\
&= p^n q^{t-n} \sum_{k=n-2}^{t-2} \binom{k}{n-2} \\
&= p^n q^{t-n} \binom{t-1}{n-1},
\end{aligned}$$

because the following identity holds

$$\sum_{k=r}^{n} \binom{k}{r} = \sum_{l=0}^{n-r} \binom{l+r}{r} = \binom{n+1}{r+1}. \qquad \square$$

Problem 14. *(Additional problem 2).*Two toy examples:

(a) Show that

$$X \sim \begin{cases} 1 & \text{w.p. } \theta \\ 2 & \text{w.p. } 2\theta \\ 3 & \text{w.p. } 1-3\theta \end{cases}$$

is not complete where $\theta \in [0, \frac{1}{3}]$.

(b) Show that

$$X \sim \begin{cases} 1 & \text{w.p. } \theta \\ 2 & \text{w.p. } \theta^2 \\ 3 & \text{w.p. } 1-\theta-\theta^2 \end{cases}$$

is complete where $\theta \in [0, \frac{1}{100}]$.

Solution of (a). We can find a non-trivial unbiased estimator for 0, for example, $\delta(X) = 3\mathbb{1}\{X = 1\} + \mathbb{1}\{X = 3\} - 1$. One can easily check that

$$\mathsf{E}_\theta\, \delta(X) = 3\,\mathsf{P}(X = 1) + \mathsf{P}(X = 3) - 1 = 0. \qquad \square$$

Solution of (b). Suppose $\mathsf{E}_\theta\, \delta(X) = 0$ for all $\theta \in [0, \frac{1}{100}]$. This implies that for all θ

$$\delta(1)\theta + \delta(2)\theta^2 + \delta(3)(1 - \theta - \theta^2) = 0,$$

or

$$(\delta(2) - \delta(3))\theta^2 + (\delta(1) - \delta(3))\theta + \delta(3) = 0.$$

This implies that $\delta(2) = \delta(3) = \delta(1) = 0$ to make the polynomial constantly 0 for a closed interval, so X is complete. □

Problem 15. *(Additional problem 3).*(Uniform problem) Let $X_1, \ldots, X_n \sim$ i.i.d. Unif$[\alpha, \beta]$. We want to estimate $g(\alpha, \beta) = \frac{\alpha+\beta}{2}$.

(a) Show that $T = (X_{(1)}, X_{(n)})$ is sufficient.
(b) Calculate $\mathsf{E}[X_1 | X_{(1)}, X_{(n)}]$.
(c) Show that the Rao-Blackwellized unbiased estimator for this uniform problem $\mathsf{E}[\overline{X} | X_{(1)}, X_{(n)}]$ is also unbiased for true mean for all symmetric distributions in $\mathcal{L}^2$.

Solution of (a). Since a single random variable follows $f_{\alpha,\beta}(x) = \frac{1}{\beta-\alpha}\mathbb{1}_{[\alpha,\beta]}(x)$, we have joint distribution of $X_1, \ldots, X_n$

$$f_{\alpha,\beta}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\beta - \alpha}\mathbb{1}_{[\alpha,\beta]}(x_i) = \prod_{i=1}^{n} \frac{1}{\beta - \alpha}\mathbb{1}_{[\alpha,\beta]}(x_{(i)}) = \frac{1}{(\beta - \alpha)^n}\mathbb{1}_{[\alpha,\beta]}(x_{(1)})\mathbb{1}_{[\alpha,\beta]}(x_{(n)}).$$

Therefore, by Neyman-Fisher factorization Theorem, $T = (X_{(1)}, X_{(n)})$ is a sufficient statistic. □

Solution of (b). Note that given T, $X_1, \ldots, X_n$ are still identically distributed, and $X_(i)|[X_{(1)}, X_{(n)}] \sim$ Unif$[X_{(1)}, X_{(n)}]$ for $2 \le i \le n - 1$. Hence,

$$\begin{aligned}
\mathsf{E}[X_1 | X_{(1)}, X_{(n)}] &= \frac{1}{n}\,\mathsf{E}[X_1 + \ldots + X_n | X_{(1)}, X_{(n)}] \\
&= \frac{1}{n}\,\mathsf{E}[X_{(1)} + \ldots + X_{(n)} | X_{(1)}, X_{(n)}] \\
&= \frac{1}{n}\left(X_{(1)} + X_{(n)} + (n-2)\frac{X_{(1)} + X_{(n)}}{2}\right) \\
&= \frac{X_{(1)} + X_{(n)}}{2}.
\end{aligned}$$

□

Solution of (c). The resulting Rao-Blackwellized unbiased estimator is

$$\mathsf{E}[\overline{X} | X_{(1)}, X_{(n)}] = \frac{X_{(1)} + X_{(n)}}{2}.$$

Now consider an arbitrary symmetric ditribution with mean θ. Then this is equivalent to that $X - \theta$ is symmetric about 0, or equivalently $X - \theta \sim \theta - X$. Therefore, we have

$$(X_{(1)} - \theta) + (X_{(n)} - \theta) \sim (\theta - X_{(1)}) + (\theta - X_{(n)}),$$

and this implies that $\mathsf{E}[(X_{(1)} - \theta) + (X_{(n)} - \theta)] = 0$. Henceforth,

$$\mathsf{E}\left[\frac{X_{(1)} + X_{(n)}}{2}\right] = \theta,$$

so $\frac{X_{(1)}+X_{(n)}}{2}$ is unbiased for θ. $\square$