MATH 281A: Homework #6

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Problem 1. *(Problem 2.1.21).*

Solution. If $X_1, \ldots, X_n \sim \text{Bern}(p)$, then $T = \sum_i X_i$ is a complete sufficient statistic. Our target is $g(p) = p^3$, and the naive guess suggested is

$$\delta(X) = \begin{cases} 1 & \text{if } X_1 = X_2 = X_3 = 1 \\ 0 & \text{o.w.} \end{cases}.$$

We Rao-Blackwellize the estimator to get the UMVUE as follows:

$$\begin{aligned}
\eta(t) &= \mathsf{E}[\delta(X) | T = t] \\
&= \mathsf{E}[\delta(X_1, \ldots, X_n) | T = t] \\
&= \mathsf{P}\left(X_1 = X_2 = X_3 = 1 \middle| \sum_i X_i = t \right) \\
&= \frac{\mathsf{P}(X_1 = X_2 = X_3 = 1, \sum_i X_i = t)}{\mathsf{P}(\sum_i X_i = t)} \\
&= \frac{\mathsf{P}(X_1 = 1)\,\mathsf{P}(X_2 = 1)\,\mathsf{P}(X_3 = 1)\,\mathsf{P}(\sum_{i=4}^n X_i = t - 3)}{\mathsf{P}(\sum_i X_i = t)} \\
&= p^3 \frac{\binom{n-3}{t-3} p^{t-3} (1-p)^{n-t}}{\binom{n}{t} p^t (1-p)^{n-t}} \\
&= \frac{\binom{n-3}{t-3}}{\binom{n}{t}} \\
&= \frac{t(t-1)(t-2)}{n(n-1)(n-2)}.
\end{aligned}$$

□

Problem 2. *(Problem 2.2.1).*

Solution. Assume $X_1, \ldots, X_n \sim$ i.i.d. $\mathcal{N}(\xi, \sigma^2)$ with σ^2 known. We know that $T = \sum_i X_i$ is a complete sufficient statsitic. Let $\overline{X} = Y + \xi \sim \mathcal{N}(\xi, \frac{\sigma^2}{n})$, so that $Y \sim \mathcal{N}(0, \frac{\sigma^2}{n})$.

Solution of (a). $\mathsf{E}[\overline{X}^2] = \mathsf{E}[(Y+\xi)^2] = \mathsf{E}\,Y^2 + 2\xi\,\mathsf{E}[Y] + \xi^2 = \frac{\sigma^2}{n} + \xi^2$. Thus,

$$\hat{\xi^2} = \overline{X}^2 - \frac{\sigma^2}{n}$$

is the UMVUE for ξ^2. □

Solution of (b). Likewise, $\mathsf{E}[\overline{X}^3] = \mathsf{E}[(Y+\xi)^3] = 3\,\mathsf{E}\,Y^2 \cdot \xi + \xi^3 = \frac{3\sigma^2}{n}\xi + \xi^3$. Thus,

$$\hat{\xi^3} = \overline{X}^3 - \frac{3\sigma^2}{n}\overline{X}$$

is the UMVUE for ξ^3. □

Problem 3. *(Problem 2.2.5).*

Solution. If $X_1, \ldots, X_n \sim$ i.i.d. $\mathcal{N}(\xi, \sigma^2)$, then $T = (\overline{X}, S^2)$ is a complete sufficient statistic. Therefore, since

$$\mathsf{E}\,\delta(X) = \mathsf{E}[\overline{X}^2] - \frac{\mathsf{E}[S^2]}{n(n-1)} = \frac{\sigma^2}{n} + \xi^2 - \frac{\sigma^2}{n} = \xi^2,$$

so δ is unbiased, a function of T, and thus is the UMVUE for ξ^2. □

Problem 4. *(Problem 2.2.7).*

Solution. Assume $X \sim N(\xi, \sigma^2)$. If an unbiased estimator δ of σ^2 exists when ξ is unknown,

$$\mathsf{E}_{\xi,\sigma^2}[\delta(X)] = \sigma^2 \quad \forall \xi, \sigma^2.$$

As the hint suggests, for fixed $\sigma = a$, X is a complete sufficient statistic for ξ, and thus $\mathsf{E}_\xi[\delta(X)] = a^2$ for all ξ implies $\delta(X) = a^2$ almost surely. However, from the uniqueness of UMVUEs, this is a contradiction. Hence, such an unbiased estimator of σ^2 does not exist. □

Problem 5. *(Problem 2.2.25).*

Solution. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be i.i.d. as $\mathrm{Unif}(0, \theta)$ and $\mathrm{Unif}(0, \theta')$, respectively. We know that $(X_{(m)}, Y_{(n)})$ is a complete sufficient statistic of the data. (Lehmann and Casella, Example 6.23.) Since

$$\frac{\theta}{2} = \mathsf{E}_\theta\left[\overline{X}\right] = \mathsf{E}_\theta\left[\frac{X_{(1)} + \ldots + X_{(m)}}{m}\right],$$

$\frac{2}{m}\sum_{i=1}^m X_{(i)}$ is an unbiased estimator of θ. Using Rao-Blackwell Theorem,

$$\hat{\theta} = \mathsf{E}_\theta\left[\frac{2}{m}(X_{(1)} + \ldots + X_{(m)})\middle| X_{(m)}\right] = \frac{2}{m}\left((m-1)\frac{X_{(m)}}{2} + X_{(m)}\right) = \frac{m+1}{m}X_{(m)}$$

is the UMVUE of θ. Now we will derive the UMVUE of $1/\theta'$. Since $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$, the cdf of $Y_{(n)}$ is

$$\mathsf{P}(Y_{(n)} \le t) = \mathsf{P}(Y_1 \le t) \cdots \mathsf{P}(Y_n \le t) = \frac{t^n}{\theta'}$$

for $t \in [0, \theta']$. Hence, $Y_{(n)}$ has pdf $f_{Y_{(n)}}(t) = \frac{nt^{n-1}}{(\theta')^n} \mathbb{1}_{t \in [0,\theta']}$. Therefore,

$$\mathsf{E}_{\theta'}\left[\frac{1}{Y_{(n)}}\right] = \int_0^{\theta'} \frac{1}{y} \frac{ny^{n-1}}{(\theta')^n} dy = \frac{n}{n-1} \frac{1}{\theta'}.$$

Hence, the UMVUE of $1/\theta'$ is

$$\widehat{\frac{1}{\theta'}} = \frac{n-1}{n} \frac{1}{Y_{(n)}}.$$

Therefore, since X^m and Y^n are independent, the UMVUE of θ/θ' is

$$\widehat{\frac{\theta}{\theta'}} = \hat{\theta} \widehat{\frac{1}{\theta'}} = \frac{(m+1)(n-1)}{mn} \frac{X_{(m)}}{Y_{(n)}}.$$

□

Problem 6. *(Problem 2.2.27).*

Solution of (a). The bias of the ML estimator $\Phi(u - \overline{X})$ is

$$\text{bias}(\xi) = \mathsf{E}[\Phi(u - \overline{X})] - \Phi(u - \xi).$$

Note that $u - \overline{X} \sim \mathcal{N}(u - \xi, \frac{1}{n})$. Therefore, if $\xi = u$, then $u - \overline{X} \sim \mathcal{N}(0, \frac{1}{n})$. Also, since $\Phi(z) - \Phi(0)$ is an odd function, we get

$$\mathsf{E}[\Phi(u - \overline{X}) - \Phi(0)] = 0,$$

which implies $\text{bias}(u) = 0$. □

Solution of (b).

$$R_{\text{ML}}(\xi) = \mathsf{E}_\xi\left[(\Phi(u - \overline{X}) - \Phi(u - \xi))^2\right], \quad R_\delta(\xi) = \mathsf{E}_\xi\left[\left(\Phi\left(\sqrt{\frac{n}{n-1}}(u - \overline{X})\right) - \Phi(u - \xi)\right)^2\right].$$

Then at $\xi = u$, the difference of the expectd square error is,

$$\begin{aligned} R_\delta(u) - R_{\text{ML}}(u) &= \mathsf{E}_{\xi=u}\left[\left(\Phi\left(\sqrt{\frac{n}{n-1}}(u - \overline{X})\right) - \Phi(0)\right)^2\right] - \mathsf{E}_{\xi=u}\left[(\Phi(u - \overline{X}) - \Phi(0))^2\right] \\ &= \mathsf{E}_{\xi=u}\left[\Phi\left(\sqrt{\frac{n}{n-1}}(u - \overline{X})\right)^2 - \Phi(u - \overline{X})^2\right]. \end{aligned}$$

However, since $\left(\sqrt{\frac{n}{n-1}}(u - \overline{X}\right)^2 > (u - \overline{X})^2$ always and $\Phi(\cdot)$ is strictly increasing, the integrand is always positive. Hence, $R_\delta(u) - R_{\text{ML}}(u) > 0$. □

Problem 7. *(Problem 2.3.18).*

Solution. Let $X \sim \text{Poisson}(\theta)$. Suppose there exists an unbiased estimator $\delta(X)$ of $1/\theta$. Then for all θ,

$$\frac{1}{\theta} = \mathsf{E}_\theta\, \delta(X) = \sum_{x=0}^{\infty} e^{-\theta} \frac{\theta^x}{x!} \delta(x),$$

and thus

$$\sum_{x=0}^{\infty} \frac{\theta^x}{x!} = \sum_{x=1}^{\infty} \delta(x-1) \frac{\theta^x}{(x-1)!}.$$

However, this does not hold in general, since the right hand side does not have a constant term. Hence, there is no such an unbiased estimator of $1/\theta$. □

Problem 8. *(Problem 2.3.20).*

Solution of (c). One can easily see that

$$\mathsf{P}_\lambda(x) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^x}{x!} \quad \text{for } x = 1, 2, \ldots.$$

Note that

$$\mathsf{E}_\lambda\, X = \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \frac{\lambda}{1-e^{-\lambda}}.$$

On the other hand, we would get

$$\begin{aligned}
\log \mathsf{P}_\lambda(x) &= -\lambda - \log(1-e^{-\lambda}) + x \log \lambda - \log x!, \\
\frac{\partial}{\partial \lambda} \log \mathsf{P}_\lambda(x) &= -\frac{1}{1-e^{-\lambda}} + \frac{x}{\lambda}, \\
\frac{\partial^2}{\partial \lambda^2} \log \mathsf{P}_\lambda(x) &= \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} - \frac{x}{\lambda^2}.
\end{aligned}$$

Therefore,

$$i(\theta) = \frac{\mathsf{E}\, X_1}{\lambda^2} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} = \frac{1}{\lambda(1-e^{-\lambda})} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} = \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})^2}.$$

Thus, the CRLB of this problem is

$$\operatorname{Var} \widehat{\lambda} \geq \frac{\lambda(1-e^{-\lambda})^2}{n(1-e^{-\lambda-\lambda e^{-\lambda}})}.$$

□

Problem 9. *(Problem 2.3.21).*

Solution. Let $Y \sim \text{Poisson}(\lambda)$ and $Z = Y | \{Y \leq a\}$. Then

$$\mathsf{P}_\lambda(Z = z) = \frac{\mathsf{P}(Y = y, Y \leq a)}{\mathsf{P}(Y \leq a)} = \frac{\lambda^y}{A(\lambda) y!} \mathbb{1}_{y \in [0,a]},$$

where $A(\lambda) = \sum_{x=0}^{a} \frac{\lambda^x}{x!}$. Suppose there exists an unbiased estimator $\delta(Z)$ of λ. Then for all $\lambda > 0$,

$$\sum_{z=0}^{a} \delta(z)\, \mathsf{P}_\lambda(Z = z) = \frac{1}{A(\lambda)} \sum_{z=0}^{a} \frac{\delta(z)}{z!} \lambda^z = \lambda.$$

After some algebra, we have

$$\sum_{z=1}^{a} \left(\frac{\delta(z)}{z!} - \frac{1}{(z-1)!} \right) \lambda^z + \delta(0) = \frac{\lambda^{a+1}}{a!} \quad \forall \lambda > 0.$$

This cannot be true, however, for any choice of $\delta(\cdot)$, because degrees of LHS and RHS differ. Hence, there exists no unbiased estimator of λ. □

Problem 10. *(Problem 2.3.23).* Suppose $X_1, \ldots, X_n \sim$ i.i.d. Poisson(λ), and consider estimation of $e^{-b\lambda}$, where b is known.

Solution of (a). $T = \sum_i X_i$ is a complete sufficient statistic. We are given $\delta^*(X) = \eta(T) = (1 - \frac{b}{n})^T$. The expectation is

$$\mathsf{E}\, \eta(T) = \sum_{x=0}^{\infty} e^{-n\lambda} \frac{(n\lambda)^x}{x!} \left(1 - \frac{b}{n}\right)^x = e^{-b\lambda} \sum_{x=0}^{\infty} e^{-(n-b)\lambda} \frac{((n-b)\lambda)^x}{x!} = e^{-b\lambda},$$

so it is unbiased. By Lehmann-Scheffe Theorem, δ^* is the UMVUE. □

Solution of (b). If $b > n$, δ^* is positive if T is even, and negative if T is odd. Therefore, its behavior is not desirable as an estimator of a positive quantity $e^{-b\lambda}$. □

Problem 11. *(Problem 1.6.29).* If a minimal sufficient statistic exists, a necessary condition for a sufficient statistic to be complete is for it to be minimal.

Solution. Suppose that $T = h(U)$ is minimal sufficient and U is complete. If U is not equivalent to T, there exists a function ψ such that $\psi(U) \neq \mathsf{E}[\psi(U)|T]$ with positive probability. However, by law of iterated expectation, we have

$$\mathsf{E}[\mathsf{E}[\psi(U)|T] - \psi(U)] = 0,$$

and thus $\mathsf{E}[\psi(U)|T] - \psi(U)$ is an unbiased estimator of 0. Now, it follows that $\mathsf{E}[\psi(U)|T] - \psi(U) = \mathsf{E}[\psi(U)|h(U)] - \psi(U) = 0$ almost surely from completeness of U, which is a contradiction. Hence, U is equivalent to the minimal sufficient statistic T. □

Problem 12. *(Problem 1.6.32).*

Solution of (a). $\mathcal{P}_0, \mathcal{P}_1$ are two families of distributions such that $\mathcal{P}_0 \subset \mathcal{P}_1$ and every null set of $\mathcal{P}_0$ is also a null set of $\mathcal{P}_1$. Assume T is complete for $\mathcal{P}_0$. Then,

$$\mathsf{E}_F[\delta(T)] = 0 \quad \forall F \in \mathcal{P}_0 \implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_0).$$

We have

$$\mathsf{E}_G[\delta(T)] = 0 \quad \forall G \in \mathcal{P}_1 \implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_0) \tag{0.1}$$
$$\implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_1), \tag{0.2}$$

so this implies T is also complete for $\mathcal{P}_1$. Note that eq. (0.1) follows from $\mathcal{P}_0 \subset \mathcal{P}_1$, and eq. (0.2) follows because every null set of $\mathcal{P}_0$ is also a null set of $\mathcal{P}_1$. □

Solution of (b). $\mathcal{P}_0 = \{\text{Binom}(n,p) \colon 0 < p < 1\}$ where n is fixed, and $\mathcal{P}_1 = \mathcal{P}_0 \cup \{\text{Poisson}(1)\}$.

$$\mathsf{E}_p\, \delta(X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(k) = 0 \quad \forall p \in (0,1) \implies \sum_{k=0}^{n} \binom{n}{k} \rho^k \delta(k) = 0 \quad \forall \rho > 0$$
$$\implies \delta(X) \equiv 0 \text{ (a.e.)}$$

Hence, $\mathcal{P}_0$ is complete. However, considering $\mathcal{P}_1$, we assume

$$\mathsf{E}_p\, \delta(X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(k) = 0 \quad \forall p \in (0,1),$$

and

$$\mathsf{E}_{\text{Poisson}(1)}\, \delta(X) = \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} \delta(k) = 0.$$

From the first restriction, it is required that $\delta(0) = \ldots = \delta(n) = 0$ as we derived. However, the second restriction

$$\sum_{k=n+1}^{\infty} \frac{\delta(k)}{k!} = 0$$

can be satisfied with a simple choice of δ, for example, $\delta(n+1) = (n+1)!$, $\delta(n+2) = -(n+2)!$, and $\delta(x) = 0$ for $x \geq n+3$. Hence, $\mathcal{P}_1$ is not complete. □

Problem 13. *(Additional problem 1).* Show that any finite family of densities on $\mathbb{R}$ with common support is an exponential family. If the family has more than one density, the parameter space is not "natural".

Solution. Let $\mathcal{F} = \{f_1(x), \ldots, f_N(x)\}$. Then we can express this family as the following form.

$$\mathcal{F} = \left\{ g_{\boldsymbol{\eta}}(x) \colon g_{\boldsymbol{\eta}}(x) = \exp\left(\sum_{i=1}^{N} \eta_i \log f_i(x) \right), \boldsymbol{\eta} \in \{e_1, \ldots, e_N\} \right\},$$

where we denote e_i as a standard unit vector. Clearly, if $N \geq 2$, then the parameter space is not natural. □

Problem 14. *(Additional problem 2).* Define $\mathsf{E}^{(\lambda)}\, X = (\mathsf{E}\, X^\lambda)^{1/\lambda}$.

(a) Show $\lim_{\lambda \to 0} \mathsf{E}^{(\lambda)} = e^{\mathsf{E} \log X}$.

(b) Extending the definition through $\lambda = 0$, show that $\mathsf{E}^{(\lambda)} X$ is monotonically increasing in λ.

Solution of (a). We want to prove

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \log \mathsf{E}\, X^\lambda = \mathsf{E} \log X.$$

Using L'Hopital's Law, it follows that

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \log \mathsf{E}\, X^\lambda = \lim_{\lambda \to 0} \frac{\frac{\mathsf{E}[X^\lambda \log X]}{\mathsf{E}[X^\lambda]}}{1} = \frac{\mathsf{E}[X^0 \log X]}{\mathsf{E}[X^0]} = \mathsf{E} \log X. \qquad \square$$

Solution of (b). Consider $\eta > \lambda > 0$. Then $x \mapsto x^{\frac{\eta}{\lambda}}$ is (strictly) convex, and thus using Jensen's Inequality, we would get

$$(\mathsf{E}\, X^\lambda)^{\frac{\eta}{\lambda}} > \mathsf{E}(X^\lambda)^{\frac{\eta}{\lambda}} = \mathsf{E}\, X^\eta,$$

which implies $\mathsf{E}^{(\eta)} X > \mathsf{E}^{(\lambda)} X$. Likewise, one can prove that it also holds when $0 > \eta > \lambda$. Since $\mathsf{E}^{(\eta)} X > \mathsf{E}^{(0)} X > \mathsf{E}^{(\lambda)} X$ for $\eta > 0 > \lambda$, $\mathsf{E}^{(\lambda)} X$ is monotonically increasing in λ. $\square$

Problem 15. *(Additional problem 3).* For a distribution symmetric with respect to its mean, show that a statistic

$$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i^3 \right)$$

is not complete.

Solution. Let us denotde $T = (T_1, T_2, T_3)$. Let $\theta := \mathsf{E}\, X_1$. Then by symmetry, we know that $\mathsf{E}(X_1 - \theta)^3 = 0$. Expanding the terms, we would get

$$\mathsf{E}(X_1 - \theta)^3 = \mathsf{E}\, X_1^3 - 3\theta\, \mathsf{E}\, X_1^2 + 3\theta^2\, \mathsf{E}\, X_1 - \theta^3 = \mathsf{E}\, X_1^3 - 3\theta\, \mathsf{E}\, X_1^2 + 2\theta^3 = 0.$$

Then we consider a function of data $\delta(X_1, \ldots, X_n) = \sum_i (X_i - \overline{X})^3$. Then

$$\begin{aligned} \sum_i (X_i - \overline{X})^3 &= \sum_i (X_i^3 - 3X_i^2 \overline{X} + 3X_i \overline{X}^2 - \overline{X}^3) \\ &= \sum_i X_i^3 - 3 \sum_i X_i^2 \overline{X} + 3n\overline{X}^3 - n\overline{X}^3 \\ &= T_3 - 3nT_1T_2 + \frac{2}{n^2} T_1^3, \end{aligned}$$

δ is a function of T. Also, we observe

$$\begin{aligned} \mathsf{E}(X_1 - \overline{X})^3 &= \mathsf{E}\, X_1^3 - 3\, \mathsf{E}\, X_1^2 \overline{X} + 3\, \mathsf{E}\, X_1 \overline{X}^2 - \mathsf{E}\, \overline{X}^3 \\ &= \mathsf{E}\, X_1^3 - \frac{3}{n} \left(\mathsf{E}\, X_1^3 + (n-1)\, \mathsf{E}\, X_1^2\, \mathsf{E}\, X_1 \right) \\ &\quad + \frac{3}{n^2} \left(\mathsf{E}\, X_1^3 + 3(n-1)\, \mathsf{E}\, X_1^2\, \mathsf{E}\, X_1 + (n-1)(n-2)(\mathsf{E}\, X_1)^3 \right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{n^3}\left(n\,\mathsf{E}\,X_1^3 + 3n(n-1)\,\mathsf{E}\,X_1^2 \cdot \mathsf{E}\,X_1 + n(n-1)(n-2)(\mathsf{E}\,X_1)^3\right) \\
&= \frac{\mathsf{E}\,X_1^3}{n^2}(n-1)(n-2) - \frac{\mathsf{E}\,X_1^2\,\mathsf{E}\,X_1}{n^2}3(n-1)(n-2) + \frac{(\mathsf{E}\,X_1)^3}{n^2}2(n-1)(n-2) \\
&= \frac{(n-1)(n-2)}{n^2}\left(\mathsf{E}\,X_1^3 - 3\,\mathsf{E}\,X_1^2\,\mathsf{E}\,X_1 + 2(\mathsf{E}\,X_1)^3\right) \\
&= \frac{(n-1)(n-2)}{n^2}\,\mathsf{E}(X_1 - \mathsf{E}\,X_1)^3 = 0.
\end{aligned}
$$

Thus, δ is a nontrivial unbiased estimator of 0. Hence, T is not a complete statistic. $\square$