Problem 1. (Problem 2.1.21).
Solution. If $X_1, \ldots, X_n \sim \text{Bern}(p)$, then $T = \sum_i X_i$ is a complete sufficient statistic. Our target is $g(p) = p^3$, and the naive guess suggested is

$$
\delta(X) = \begin{cases} 
1 & \text{if } X_1 = X_2 = X_3 = 1 \\
0 & \text{o.w.}
\end{cases}
$$

We Rao-Blackwellize the estimator to get the UMVUE as follows:

$$
\eta(t) = \mathbb{E}[\delta(X) | T = t] \\
= \mathbb{E}[\delta(X_1, \ldots, X_n) | T = t] \\
= \mathbb{P}
\left(
X_1 = X_2 = X_3 = 1 \mid \sum_i X_i = t
\right) \\
= \frac{\mathbb{P}(X_1 = X_2 = X_3 = 1, \sum_i X_i = t)}{\mathbb{P}(\sum_i X_i = t)} \\
= \frac{\mathbb{P}(X_1 = 1) \mathbb{P}(X_2 = 1) \mathbb{P}(X_3 = 1) \mathbb{P}(\sum_{i=4}^n X_i = t - 3)}{\mathbb{P}(\sum_i X_i = t)} \\
= p^3 \binom{n-3}{t-3} p^{t-3} (1-p)^{n-t} \\
= \frac{\binom{n-3}{t-3} p^t (1-p)^{n-t}}{\binom{n}{t}} \\
= \frac{t(t-1)(t-2)}{n(n-1)(n-2)}.
$$

Problem 2. (Problem 2.2.1).
Solution. Assume $X_1, \ldots, X_n \sim \text{i.i.d. } \mathcal{N}(\xi, \sigma^2)$ with $\sigma^2$ known. We know that $T = \sum_i X_i$ is a complete sufficient statistic. Let $\overline{X} = Y + \xi \sim \mathcal{N}(\xi, \sigma^2/n)$, so that $Y \sim \mathcal{N}(0, \sigma^2/n)$. 

Solution of (a). \( E[X^2] = E[(Y + \xi)^2] = EY^2 + 2\xi E[Y] + \xi^2 = \frac{\sigma^2}{n} + \xi^2. \) Thus,
\[
\hat{\xi}^2 = X^2 - \frac{\sigma^2}{n}
\]
is the UMVUE for \( \xi^2. \)

Solution of (b). Likewise, \( E[X^3] = E[(Y + \xi)^3] = 3EY^2 \cdot \xi + \xi^3 = \frac{3\sigma^2}{n} \xi + \xi^3. \) Thus,
\[
\hat{\xi}^3 = X^3 - \frac{3\sigma^2}{n} X
\]
is the UMVUE for \( \xi^3. \)

Problem 3. (Problem 2.2.5).
Solution. If \( X_1, \ldots, X_n \sim \text{i.i.d. } \mathcal{N}(\xi, \sigma^2), \) then \( T = (X, S^2) \) is a complete sufficient statistic. Therefore, since
\[
E \delta(X) = E[X^2] - \frac{E[S^2]}{n(n-1)} = \frac{\sigma^2}{n} + \xi^2 - \frac{\sigma^2}{n} = \xi^2,
\]
so \( \delta \) is unbiased, a function of \( T, \) and thus is the UMVUE for \( \xi^2. \)

Problem 4. (Problem 2.2.7).
Solution. Assume \( X \sim \mathcal{N}(\xi, \sigma^2). \) If an unbiased estimator \( \delta \) of \( \sigma^2 \) exists when \( \xi \) is unknown,
\[
E_{\xi,\sigma^2}[\delta(X)] = \sigma^2 \forall \xi, \sigma^2.
\]
As the hint suggests, for fixed \( \sigma = a, \) \( X \) is a complete sufficient statistic for \( \xi, \) and thus \( E_{\xi}[\delta(X)] = a^2 \) for all \( \xi \) implies \( \delta(X) = a^2 \) almost surely. However, from the uniqueness of UMVUEs, this is a contradiction. Hence, such an unbiased estimator of \( \sigma^2 \) does not exist.

Problem 5. (Problem 2.2.25).
Solution. Let \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) be i.i.d. as \( \text{Unif}(0, \theta) \) and \( \text{Unif}(0, \theta'), \) respectively. We know that \( (X_{(m)}, Y_{(n)}) \) is a complete sufficient statistic of the data. (Lehmann and Casella, Example 6.23.) Since
\[
\frac{\theta}{2} = E_{\theta}[X] = E_{\theta} \left[ \frac{X(1) + \ldots + X(m)}{m} \right],
\]
\[
\frac{2}{m} \sum_{i=1}^{m} X_{(i)} \text{ is an unbiased estimator of } \theta. \] Using Rao-Blackwell Theorem,
\[
\hat{\theta} = E_{\theta} \left[ \frac{2}{m} (X(1) + \ldots + X(m)) \middle| X(m) \right] = \frac{2}{m} \left( (m - 1) \frac{X(m)}{2} + X(m) \right) = \frac{m+1}{m} X(m)
\]
is the UMVUE of \( \theta. \) Now we will derive the UMVUE of \( 1/\theta'. \) Since \( Y_{(n)} = \max\{Y_1, \ldots, Y_n\}, \) the cdf of \( Y_{(n)} \) is
\[
P(Y_{(n)} \leq t) = P(Y_1 \leq t) \cdot \ldots \cdot P(Y_n \leq t) = \frac{t^n}{\theta'}
\]
for $t \in [0, \theta']$. Hence, $Y(n)$ has pdf $f_{Y(n)}(t) = \frac{n t^{n-1}}{(\theta')^n} \mathbb{1}_{t \in [0, \theta']}$. Therefore,

$$E_{\theta'} \left[ \frac{1}{Y(n)} \right] = \int_0^{\theta'} \frac{1}{y} \left( \frac{n y^{n-1}}{(\theta')^n} \right) dy = \frac{n}{n - 1} \frac{1}{\theta'}.$$ 

Hence, the UMVUE of $1/\theta'$ is

$$\hat{1}/\theta' = \frac{n - 1}{n} \frac{1}{Y(n)}.$$ 

Therefore, since $X^m$ and $Y^n$ are independent, the UMVUE of $\theta/\theta'$ is

$$\hat{\theta}/\theta' = \hat{1}/\theta' = \frac{(m + 1)(n - 1)}{mn} \frac{X(m)}{Y(n)}.$$ 

\textbf{Problem 6.} (Problem 2.2.27).

\textit{Solution of (a).} The bias of the ML estimator $\Phi(u - \bar{X})$ is

$$\text{bias}(\xi) = E[\Phi(u - \bar{X})] - \Phi(u - \xi).$$

Note that $u - \bar{X} \sim \mathcal{N}(u - \xi, \frac{1}{n})$. Therefore, if $\xi = u$, then $u - \bar{X} \sim \mathcal{N}(0, \frac{1}{n})$. Also, since $\Phi(z) - \Phi(0)$ is an odd function, we get

$$E[\Phi(u - \bar{X}) - \Phi(0)] = 0,$$

which implies $\text{bias}(u) = 0$. \hfill \Box$

\textit{Solution of (b).}

$$R_{\text{ML}}(\xi) = E_{\bar{X}} \left[ (\Phi(u - \bar{X}) - \Phi(u - \xi))^2 \right], \quad R_{\delta}(\xi) = E_{\xi} \left[ \left( \Phi \left( \sqrt{\frac{n}{n - 1}} (u - \bar{X}) \right) - \Phi(u - \xi) \right)^2 \right].$$

Then at $\xi = u$, the difference of the expected square error is,

$$R_{\delta}(u) - R_{\text{ML}}(u) = E_{\xi=u} \left[ \left( \Phi \left( \sqrt{\frac{n}{n - 1}} (u - \bar{X}) \right) - \Phi(0) \right)^2 \right] - E_{\xi=u} \left[ (\Phi(u - \bar{X}) - \Phi(0))^2 \right]$$

$$= E_{\xi=u} \left[ \Phi \left( \sqrt{\frac{n}{n - 1}} (u - \bar{X}) \right)^2 - \Phi(u - \bar{X})^2 \right].$$

However, since $\left( \sqrt{\frac{n}{n - 1}} (u - \bar{X}) \right)^2 > (u - \bar{X})^2$ always and $\Phi(\cdot)$ is strictly increasing, the integrand is always positive. Hence, $R_{\delta}(u) - R_{\text{ML}}(u) > 0$. \hfill \Box$

\textbf{Problem 7.} (Problem 2.3.18).
Solution. Let $X \sim \text{Poisson}(\theta)$. Suppose there exists an unbiased estimator $\delta(X)$ of $1/\theta$. Then for all $\theta$,

$$\frac{1}{\theta} = E_\theta \delta(X) = \sum_{x=0}^{\infty} e^{-\theta} \frac{\theta^x}{x!} \delta(x),$$

and thus

$$\sum_{x=0}^{\infty} \frac{\theta^x}{x!} = \sum_{x=1}^{\infty} \frac{\delta(x-1)}{(x-1)!} \theta^x.$$

However, this does not hold in general, since the right hand side does not have a constant term. Hence, there is no such an unbiased estimator of $1/\theta$. \hfill \Box

**Problem 8.** (Problem 2.3.20).

**Solution of (c).** One can easily see that

$$P_\lambda(x) = \frac{e^{-\lambda} \lambda^x}{1 - e^{-\lambda} x!} \quad \text{for } x = 1, 2, \ldots.$$

Note that

$$E_\lambda X = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \frac{\lambda}{1 - e^{-\lambda}}.$$

On the other hand, we would get

$$\log P_\lambda(x) = -\lambda - \log(1 - e^{-\lambda}) + x \log \lambda - \log x!,$$

$$\frac{\partial}{\partial \lambda} \log P_\lambda(x) = -\frac{1}{1 - e^{-\lambda}} + \frac{x}{\lambda},$$

$$\frac{\partial^2}{\partial \lambda^2} \log P_\lambda(x) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} - \frac{x}{\lambda^2}.$$

Therefore,

$$i(\theta) = \frac{E X_1}{\lambda^2} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} = \frac{1}{\lambda(1 - e^{-\lambda})} - \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})^2}.$$

Thus, the CRLB of this problem is

$$\text{Var} \hat{\lambda} \geq \frac{\lambda(1 - e^{-\lambda})^2}{n(1 - e^{-\lambda} - \lambda e^{-\lambda})}. \hfill \Box$$

**Problem 9.** (Problem 2.3.21).

**Solution.** Let $Y \sim \text{Poisson}(\lambda)$ and $Z = Y|\{Y \leq a\}$. Then

$$P_\lambda(Z = z) = \frac{P(Y = y, Y \leq a)}{P(Y \leq a)} = \frac{\lambda^y}{A(\lambda)y!} I_{y \in [0, a]},$$
where \( A(\lambda) = \sum_{x=0}^{\lambda} \frac{\lambda^x}{x!} \). Suppose there exists an unbiased estimator \( \delta(Z) \) of \( \lambda \). Then for all \( \lambda > 0 \),

\[
\sum_{z=0}^{\lambda} \delta(z) \mathbb{P}(Z = z) = \frac{1}{A(\lambda)} \sum_{z=0}^{\lambda} \frac{\delta(z)}{z!} \lambda^z = \lambda.
\]

After some algebra, we have

\[
\sum_{z=1}^{\lambda} \left( \frac{\delta(z)}{z!} - \frac{1}{(z-1)!} \right) \lambda^z + \delta(0) = \frac{\lambda^{a+1}}{a!} \quad \forall \lambda > 0.
\]

This cannot be true, however, for any choice of \( \delta(\cdot) \), because degrees of LHS and RHS differ. Hence, there exists no unbiased estimator of \( \lambda \).

**Problem 10.** (Problem 2.3.23). Suppose \( X_1, \ldots, X_n \sim \text{i.i.d. Poisson}(\lambda) \), and consider estimation of \( e^{-b\lambda} \), where \( b \) is known.

**Solution of (a).** \( T = \sum_i X_i \) is a complete sufficient statistic. We are given \( \delta^*(X) = \eta(T) = (1 - \frac{b}{n})^T \). The expectation is

\[
\mathbb{E} \eta(T) = \sum_{x=0}^{\infty} e^{-n\lambda} \frac{(n\lambda)^x}{x!} \left( 1 - \frac{b}{n} \right)^x = e^{-b\lambda} \sum_{x=0}^{\infty} e^{-(n-b)\lambda} \frac{(n-b)\lambda^x}{x!} = e^{-b\lambda},
\]

so it is unbiased. By Lehmann-Scheffe Theorem, \( \delta^* \) is the UMVUE.

**Solution of (b).** If \( b > n \), \( \delta^* \) is positive if \( T \) is even, and negative if \( T \) is odd. Therefore, its behavior is not desirable as an estimator of a positive quantity \( e^{-b\lambda} \).

**Problem 11.** (Problem 1.6.29). If a minimal sufficient statistic exists, a necessary condition for a sufficient statistic to be complete is for it to be minimal.

**Solution.** Suppose that \( T = h(U) \) is minimal sufficient and \( U \) is complete. If \( U \) is not equivalent to \( T \), there exists a function \( \psi \) such that \( \psi(U) \neq \mathbb{E}[\psi(U)|T] \) with positive probability. However, by law of iterated expectation, we have

\[
\mathbb{E} \left[ \mathbb{E}[\psi(U)|T] - \psi(U) \right] = 0,
\]

and thus \( \mathbb{E}[\psi(U)|T] - \psi(U) \) is an unbiased estimator of 0. Now, it follows that \( \mathbb{E}[\psi(U)|T] - \psi(U) = \mathbb{E}[\psi(U)|h(U)] - \psi(U) = 0 \) almost surely from completeness of \( U \), which is a contradiction. Hence, \( U \) is equivalent to the minimal sufficient statistic \( T \).

**Problem 12.** (Problem 1.6.32).

**Solution of (a).** \( \mathcal{P}_0, \mathcal{P}_1 \) are two families of distributions such that \( \mathcal{P}_0 \subset \mathcal{P}_1 \) and every null set of \( \mathcal{P}_0 \) is also a null set of \( \mathcal{P}_1 \). Assume \( T \) is complete for \( \mathcal{P}_0 \). Then,

\[
\mathbb{E}_F[\delta(T)] = 0 \quad \forall F \in \mathcal{P}_0 \implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_0 \).
We have

\[ \mathbb{E}_G[\delta(T)] = 0 \quad \forall G \in \mathcal{P}_1 \implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_0) \]

\[ \implies \delta \equiv 0 \text{ (a.e. } \mathcal{P}_1), \]

so this implies \( T \) is also complete for \( \mathcal{P}_1 \). Note that eq. (0.1) follows from \( \mathcal{P}_0 \subset \mathcal{P}_1 \), and eq. (0.2) follows because every null set of \( \mathcal{P}_0 \) is also a null set of \( \mathcal{P}_1 \).

**Solution of (b).** \( \mathcal{P}_0 = \{ \text{Binom}(n, p) : 0 < p < 1 \} \) where \( n \) is fixed, and \( \mathcal{P}_1 = \mathcal{P}_0 \cup \{ \text{Poisson}(1) \} \).

\[ \mathbb{E}_p \delta(X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(k) = 0 \quad \forall p \in (0, 1) \implies \sum_{k=0}^{n} \binom{n}{k} \rho^k \delta(k) = 0 \quad \forall \rho > 0 \]

\[ \implies \delta(X) \equiv 0 \text{ (a.e.)} \]

Hence, \( \mathcal{P}_0 \) is complete. However, considering \( \mathcal{P}_1 \), we assume

\[ \mathbb{E}_p \delta(X) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta(k) = 0 \quad \forall p \in (0, 1), \]

and

\[ \mathbb{E}_{\text{Poisson}(1)} \delta(X) = \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} \delta(k) = 0. \]

From the first restriction, it is required that \( \delta(0) = \ldots = \delta(n) = 0 \) as we derived. However, the second restriction

\[ \sum_{k=n+1}^{\infty} \frac{\delta(k)}{k!} = 0 \]

can be satisfied with a simple choice of \( \delta \), for example, \( \delta(n+1) = (n+1)! \), \( \delta(n+2) = -(n+2)! \), and \( \delta(x) = 0 \) for \( x \geq n + 3 \). Hence, \( \mathcal{P}_1 \) is not complete.

**Problem 13. (Additional problem 1).** Show that any finite family of densities on \( \mathbb{R} \) with common support is an exponential family. If the family has more than one density, the parameter space is not “natural”.

**Solution.** Let \( \mathcal{F} = \{ f_1(x), \ldots, f_N(x) \} \). Then we can express this family as the following form.

\[ \mathcal{F} = \left\{ g_{\eta}(x) : g_{\eta}(x) = \exp \left( \sum_{i=1}^{N} \eta_i \log f_i(x) \right), \eta \in \{ e_1, \ldots, e_N \} \right\}, \]

where we denote \( e_i \) as a standard unit vector. Clearly, if \( N \geq 2 \), then the parameter space is not natural.

**Problem 14. (Additional problem 2).** Define \( \mathbb{E}^{(\lambda)} X = (\mathbb{E} X^\lambda)^{1/\lambda} \).

(a) Show \( \lim_{\lambda \to 0} \mathbb{E}^{(\lambda)} = e^{\mathbb{E} \log X} \).
(b) Extending the definition through \( \lambda = 0 \), show that \( E^{(\lambda)} X \) is monotonically increasing in \( \lambda \).

Solution of (a). We want to prove
\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \log E X^{\lambda} = E \log X.
\]
Using L'Hopital's Law, it follows that
\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \log E X^{\lambda} = \lim_{\lambda \to 0} \frac{E[X^{\lambda} \log X]}{E[X^\lambda]} = \frac{E[X^0 \log X]}{E[X^0]} = E \log X.
\]

Solution of (b). Consider \( \eta > \lambda > 0 \). Then \( x \mapsto x^{\frac{\eta}{\lambda}} \) is (strictly) convex, and thus using Jensen's Inequality, we would get
\[
(E X^{\lambda})^{\frac{\eta}{\lambda}} > E (X^{\lambda})^{\frac{\eta}{\lambda}} = E X^{\eta},
\]
which implies \( E^{(\eta)} X > E^{(\lambda)} X \). Likewise, one can prove that it also holds when \( 0 > \eta > \lambda \). Since \( E^{(\eta)} X > E^{(0)} X > E^{(\lambda)} X \) for \( \eta > 0 > \lambda \), \( E^{(\lambda)} X \) is monotonically increasing in \( \lambda \).

**Problem 15.** (Additional problem 3). For a distribution symmetric with respect to its mean, show that a statistic
\[
T = \left( \sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} X_i^3 \right)
\]
is not complete.

Solution. Let us denote \( T = (T_1, T_2, T_3) \). Let \( \theta := E X_1 \). Then by symmetry, we know that \( E(X_1 - \theta)^3 = 0 \). Expanding the terms, we would get
\[
E(X_1 - \theta)^3 = E X_1^3 - 3\theta E X_1^2 + 3\theta^2 E X_1 - \theta^3 = E X_1^3 - 3\theta E X_1^2 + 2\theta^3 = 0.
\]
Then we consider a function of data \( \delta(X_1, \ldots, X_n) = \sum_i (X_i - \bar{X})^3 \). Then
\[
\sum_i (X_i - \bar{X})^3 = \sum_i (X_i^3 - 3X_i^2 \bar{X} + 3X_i \bar{X}^2 - \bar{X}^3)
\]
\[
= \sum_i X_i^3 - 3 \sum_i X_i^2 \bar{X} + 3n \bar{X}^2 - n \bar{X}^3
\]
\[
= T_3 - 3nT_1T_2 + \frac{2}{n^2} T_1^3,
\]
\( \delta \) is a function of \( T \). Also, we observe
\[
E(X_1 - \bar{X})^3 = E X_1^3 - 3 E X_1^2 \bar{X} + 3 E X_1 \bar{X}^2 - E \bar{X}^3
\]
\[
= E X_1^3 - \frac{3}{n} (E X_1^3 + (n-1) E X_1^2 E X_1)
\]
\[
+ \frac{3}{n^2} (E X_1^3 + 3(n-1) E X_1^2 E X_1 + (n-1)(n-2)(E X_1)^3)
\]

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\[
- \frac{1}{n^3} \left(n \mathbb{E} X_1^3 + 3n(n-1) \mathbb{E} X_1^2 \cdot \mathbb{E} X_1 + n(n-1)(n-2)(\mathbb{E} X_1)^3\right)
\]
\[
= \frac{\mathbb{E} X_1^3}{n^2} (n-1)(n-2) - \frac{\mathbb{E} X_1^2 \mathbb{E} X_1}{n^2} 3(n-1)(n-2) + \frac{(\mathbb{E} X_1)^3}{n^2} - 2(n-1)(n-2)
\]
\[
= \frac{(n-1)(n-2)}{n^2} \left(\mathbb{E} X_1^3 - 3 \mathbb{E} X_1^2 \mathbb{E} X_1 + 2(\mathbb{E} X_1)^3\right)
\]
\[
= \frac{(n-1)(n-2)}{n^2} \mathbb{E}(X_1 - \mathbb{E} X_1)^3 = 0.
\]

Thus, \( \delta \) is a nontrivial unbiased estimator of 0. Hence, \( T \) is not a complete statistic. \( \square \)