Solution to Problem 1. If \( x \notin \limsup_{n \to \infty} E_n \), then \( x \in E_n \) for only finitely many \( n \), that is \( \exists k \in \mathbb{N} \) such that for any \( n \in \mathbb{N} \) that \( k \leq n < \infty \), \( x \notin E_n \), then for such \( k \), \( x \in \bigcap_{n=k}^{\infty} E_n^c \), so \( x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n^c = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n^c \), verse vesa. Hence \( \limsup_{n \to \infty} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \).

If \( x \in \liminf_{n \to \infty} E_n \), then \( x \in E_n \) for all but finitely many \( n \), that is \( \exists k \in \mathbb{N} \) such that for any \( n \in \mathbb{N} \) that \( k \leq n < \infty \), \( x \in E_n \), then for such \( k \), \( x \in \bigcap_{n=k}^{\infty} E_n \), so \( x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \), verse vesa. Hence \( \liminf_{n \to \infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \). \( \square \)

Solution #1 to Problem 3. In my solutions I always set \( \mathbb{N} = \{ n \in \mathbb{Z} \mid n > 0 \} \).
Define
\[
g: \mathbb{N}^2 \to \mathbb{N} \\
(a, b) \mapsto \frac{(a + b - 2)(a + b - 1)}{2} + b.
\]
It is not hard to show that \( g \) is bijective. Then \( f = g^{-1} \) is the required bijection. \( \square \)

Solution #2 to Problem 3 (contributed by E. Lybrand). Define
\[
g: \mathbb{N}^2 \to \mathbb{N} \\
(p, j) \mapsto (\text{the } j\text{-th minimal number in } M_p) - 1
\]
where \( M_p := \{ m > 1 \mid \min \{ q \text{ prime}, q \mid m \} \geq p \} \supset \{ p^k \}_{k \in \mathbb{N}} \) for each prime \( p \) is countable, and \( g \) is bijective. Then \( f = g^{-1} \) is the required bijection. \( \square \)
Solution #3 to Problem 3 (contributed by B. Maisel, Y. Huang, and so on). Define

\[ g : \mathbb{N}^2 \to \mathbb{N} \]
\[ (a, b) \mapsto 2(a - 1) \cdot b \]

\( g \) is bijective. Then \( f = g^{-1} \) is the required bijection.

Solution #4 to Problem 3 (contributed by O. Delight). Define

\[ f : \mathbb{N} \to \mathbb{N}^2 \]
\[ (x_k \cdots x_1 x_0)_b \mapsto \left( (x_{2\left\lfloor \frac{k}{2} \right\rfloor} \cdots x_2 x_0)_b, (x_{2\left\lfloor \frac{k+1}{2} \right\rfloor - 1} \cdots x_3 x_1)_b \right) \]

where \((x_k \cdots x_1 x_0)_b\) is the representation of natural numbers in any fixed base \(b \geq 2\). Then \( f \) is the required bijection.

Solution #1 to Problem 6. Assume \( \mathcal{A} \) is a countably infinite \( \sigma \)-algebra on \( X \). For each \( x \in X \),

\[ A_x := \left( \bigcap_{x \in A \in \mathcal{A}} A \right) \cap \left( \bigcap_{x \not\in A \in \mathcal{A}} A^c \right) \]
\[ = \left\{ y \in A \mid \forall A \in \mathcal{A}, \{x, y\} \subseteq \text{either } A \text{ or } A^c \right\}. \]

Since the intersection is actually countable, \( A_x \in \mathcal{A} \), and for any \( x, y \in X \), either \( A_x = A_y \) or \( A_x \cap A_y = \emptyset \). So \( A = \bigcup_{x \in A} A_x \) for \( A \in \mathcal{A} \). Define

\[ B := \{ A_x \mid x \in X \} \subseteq \mathcal{A}. \]

It can be seen that \( \mathcal{A} = \mathcal{M}(B) = \{ \bigcup B' \mid B' \subseteq B \} \), for \( B \) a collection of pairwise disjoint subsets of \( X \), the ‘\( \subseteq \)’ direction coming from the definition of \( A_x \). If \( B \) is finite, \( \mathcal{A} \) is finite, otherwise \( \mathcal{A} \) is uncountably infinite, for \( \text{Card}(\mathcal{A}) = \text{Card}(\mathcal{P}(B)) \). Contradiction!

Solution #2 to Problem 6 (contributed by Y. Fu and several else). Assume \( \mathcal{A} \) is a countably infinite \( \sigma \)-algebra on \( X \). For each \( x \in X \),

\[ A_x := \bigcap_{x \in A \in \mathcal{A}} A \]

Since the intersection is actually countable, \( A_x \in \mathcal{A} \). For any \( x, y \in X \), if \( A_x \cap A_y \neq \emptyset \) and \( x \notin A_y \), then \( A_x \setminus A_y \neq \emptyset \) in \( \mathcal{A} \) is a proper subset of \( A_x \) containing \( x \), which contradicts the minimal property of \( A_x \). So \( x \in A_y \) and moreover \( A_x \subset A_y \), analogously \( A_y \subset A_x \), that is \( A_x = A_y \). Thus we have

\[ A_x = \left\{ y \in A \mid \forall A \in \mathcal{A}, \{x, y\} \subseteq \text{either } A \text{ or } A^c \right\}. \]

The rest of the proof is the same as Solution #1.
Solution #3 to Problem 6 (contributed by D. Lenz). Assume $\mathcal{A}$ is a countably infinite $\sigma$-algebra on $X$. Recall that if $\mathcal{A}$ is a $\sigma$-algebra on $X$ and $\emptyset \neq Z \subset X$ then $\mathcal{A}_Z := \{ A \cap Z \mid A \in \mathcal{A} \}$ is a $\sigma$-algebra on $Z$.

Denote $X_0 = X$. For each $i \geq 1$, assuming $X_{i-1} \neq \emptyset$ and $\mathcal{A}_{X_{i-1}}$ is infinite, let nonempty $X_i, Y_i = X_{i-1} \setminus X_i \in \mathcal{A}_{X_{i-1}} \subset \mathcal{A}$, then $(X_i, \mathcal{A}_{X_i})$ and $(Y_i, \mathcal{A}_{Y_i})$ are measurable space. Without loss of generality, we can assume $\mathcal{A}_{X_i}$ to be infinite, since any nonempty $A \in \mathcal{A}_{X_{i-1}}$, at least one of $A \cap X_i$ and $A \cap Y_i$ is nonempty.

Inductively, we get a decreasing sequence $X = X_0 \supset X_1 \supset \cdots$ and countable collection of nonempty sets $\{ Y_i \mid i \in \mathbb{N} \} \subset \mathcal{A}$. Then as before $\text{Card}(\mathcal{A}) \geq \text{Card}(\mathcal{P}(\mathbb{N}))$. \qed