Math 240A: Real Analysis, Fall 2015

Solution to Selected Problems of Homework 1

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Solution to Problem 1. If $x \notin \limsup_{n\to\infty} E_n$, then $x \in E_n$ for only finitely many n, that is $\exists k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ that $k \le n < \infty$, $x \notin E_n$, then for such k, $x \in \bigcap_{n=k}^{\infty} E_n^{\complement}$, so $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n^{\complement} = \left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \right)^{\complement}$, *verse vesa.* Hence $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

If $x \in \liminf_{n\to\infty} E_n$, then $x \in E_n$ for all but finitely many n, that is $\exists k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ that $k \le n < \infty$, $x \in E_n$, then for such k, $x \in \bigcap_{n=k}^{\infty} E_n$, so $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$, *verse vesa.* Hence $\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$. $\square$

Solution #1 to Problem 3. In my solutions I always set $\mathbb{N} = \{ n \in \mathbb{Z} \mid n > 0 \}$.

Define

$$\begin{aligned} g \colon \mathbb{N}^2 &\to \mathbb{N} \\ (a,b) &\mapsto \frac{(a+b-2)(a+b-1)}{2} + b. \end{aligned}$$

It is not hard to show that g is bijective. Then $f = g^{-1}$ is the required bijection. $\square$

Solution #2 to Problem 3 (contributed by E. Lybrand). Define

$$\begin{aligned} g \colon \mathbb{N}^2 &\to \mathbb{N} \\ (p,j) &\mapsto (\text{the } j\text{-th minimal number in } M_p) - 1 \end{aligned}$$

where $M_p := \{ m > 1 \mid \min\{ q \text{ prime}, q \mid m \} \ge p \} \supset \{p^k\}_{k\in\mathbb{N}}$ for each prime p is countable, and g is bijective. Then $f = g^{-1}$ is the required bijection. $\square$

Solution #3 to Problem 3 (contributed by B. Maisel, Y. Huang, and so on). Define

$$\begin{aligned} g\colon \mathbb{N}^2 &\to \mathbb{N} \\ (a,b) &\mapsto 2^(a-1) * b \end{aligned}$$

g is bijective. Then $f = g^{-1}$ is the required bijection. $\square$

Solution #4 to Problem 3 (contributed by O. Delight). Define

$$\begin{aligned} f\colon \mathbb{N} &\to \mathbb{N}^2 \\ (x_k \cdots x_1 x_0)_b &\mapsto \left(\left(x_{2\lfloor \frac{k}{2} \rfloor} \cdots x_2 x_0 \right)_b, \left(x_{2\lfloor \frac{k+1}{2} \rfloor - 1} \cdots x_3, x_1 \right)_b \right) \end{aligned}$$

where $(x_k \cdots x_1 x_0)_b$ is the representation of natural numbers in any fixed base $b \geq 2$. Then f is the required bijection. $\square$

Solution #1 to Problem 6. Assume $\mathcal{A}$ is a countably infinite σ-algebra on X. For each $x \in X$,

$$\begin{aligned} A_x &:= \left(\bigcap_{x \in A \in \mathcal{A}} A \right) \cap \left(\bigcap_{x \notin A \in \mathcal{A}} A^{\complement} \right) \\ &= \left\{ y \in A \;\middle|\; \forall A \in \mathcal{A}, \{x,y\} \subset \text{either } A \text{ or } A^{\complement} \right\}. \end{aligned}$$

Since the intersection is actually countable, $A_x \in \mathcal{A}$, and for any $x, y \in X$, either $A_x = A_y$ or $A_x \cap A_y = \emptyset$. So $A = \cup_{x \in A} A_x$ for $A \in \mathcal{A}$. Define

$$\mathcal{B} := \{ A_x \mid x \in X \} \subset \mathcal{A}.$$

It can be seen that $\mathcal{A} = \mathcal{M}(\mathcal{B}) = \{ \cup \mathcal{B}' \mid \mathcal{B}' \subset \mathcal{B} \}$, for $\mathcal{B}$ a collection of pairwise disjoint subsets of X, the '$\subset$' direction coming from the definition of A_x. If $\mathcal{B}$ is finite, $\mathcal{A}$ is finite, otherwise $\mathcal{A}$ is uncountably infinite, for $\mathrm{Card}(\mathcal{A}) = \mathrm{Card}(\mathcal{P}(\mathcal{B}))$. Contradiction! $\square$

Solution #2 to Problem 6 (contributed by Y. Fu and several else). Assume $\mathcal{A}$ is a countably infinite σ-algebra on X. For each $x \in X$,

$$A_x := \bigcap_{x \in A \in \mathcal{A}} A$$

Since the intersection is actually countable, $A_x \in \mathcal{A}$. For any $x, y \in X$, if $A_x \cap A_y \neq \emptyset$ and $x \notin A_y$, then $A_x \setminus A_y \neq \emptyset$ in $\mathcal{A}$ is a proper subset of A_x containing x, which contradicts the minimal property of A_x. So $x \in A_y$ and moreover $A_x \subset A_y$, analogously $A_y \subset A_x$, that is $A_x = A_y$. Thus we have

$$A_x = \left\{ y \in A \;\middle|\; \forall A \in \mathcal{A}, \{x,y\} \subset \text{either } A \text{ or } A^{\complement} \right\}.$$

The rest of the proof is the same as Solution #1. $\square$

Solution #3 to Problem 6 (contributed by D. Lenz). Assume $\mathcal{A}$ is a countably infinite σ-algebra on X. Recall that if $\mathcal{A}$ is a σ-algebra on X and $\emptyset \neq Z \subset X$ then $\mathcal{A}_Z := \{ A \cap Z \mid A \in \mathcal{A} \}$ is a σ-algebra on Z.

Denote $X_0 = X$. For each $i \geq 1$, assuming $X_{i-1} \neq \emptyset$ and $\mathcal{A}_{X_{i-1}}$ is infinite, let nonempty $X_i, Y_i = X_{i-1} \setminus X_i \in \mathcal{A}_{X_{i-1}} \subset \mathcal{A}$, then $(X_i, \mathcal{A}_{X_i})$ and $(Y_i, \mathcal{A}_{Y_i})$ are measurable space. Without loss of generality, we can assume $\mathcal{A}_{X_i}$ to be infinite, since any nonempty $A \in \mathcal{A}_{X_{i-1}}$, at least one of $A \cap X_i$ and $A \cap Y_i$ is nonempty.

Inductively, we get a decreasing sequence $X = X_0 \supset X_1 \supset \cdots$ and countable collection of nonempty sets $\{ Y_i \mid i \in \mathbb{N} \} \subset \mathcal{A}$. Then as before $\mathrm{Card}(\mathcal{A}) \geq \mathrm{Card}(\mathcal{P}(\mathbb{N}))$. $\square$