Solution to Practice Midterm Exam 2

1 $F(x,y,z) = (x^2, yx^2, z+zx)$, $\vec{c}: (-\infty, 1) \to \mathbb{R}^3$, $c(t) = \left(\frac{1}{1-t}, 0, \frac{e^t}{1-t}\right)$

(1) $\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & yx^2 & z+zx \end{vmatrix} = (0, -z, 2xy)$

$\text{div } \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(yx^2) + \frac{\partial}{\partial z}(z+zx) = 2x + x^2 + 1 + x = x^2 + 3x + 1$.

(2) $\vec{F}(c(t)) = \left(\frac{1}{(1-t)^2}, 0, \frac{te^t}{(1-t)^2}\right)$.

$\vec{c}'(t) = \left(\frac{1}{(1-t)^2}, 0, \frac{te^t}{(1-t)^2}\right)$.

$\vec{F}(\vec{c}(t)) = \vec{c}'(t)$, so $\vec{c}$ is a flow line of $\vec{F}$.

2. $x^2+y^2=(e^{\frac{z}{2}}+e^{-\frac{z}{2}})^2$, $-5 \le z \le 5$.

- Find the parametrization :

let $r=e^{\frac{z}{2}}+e^{-\frac{z}{2}}$, then $x^2+y^2=r^2$, so set $x=r\cos\theta, y=r\sin\theta$

$$\Phi : [-5,5]\times[0,2\pi], \quad \Phi(z,\theta)=\begin{pmatrix} x(z,\theta) \\ y(z,\theta) \\ z(z,\theta) \end{pmatrix}=\begin{pmatrix} (e^{\frac{z}{2}}+e^{-\frac{z}{2}})\cos\theta \\ (e^{\frac{z}{2}}+e^{-\frac{z}{2}})\sin\theta \\ z \end{pmatrix}$$

- Compute $\|T_z \times T_\theta\|$

$$T_z=\begin{pmatrix} +\frac{1}{2}(e^{\frac{z}{2}}-e^{-\frac{z}{2}})\,\overset{\cos\theta}{\cancel{\sin\theta}} \\ \frac{1}{2}(e^{\frac{z}{2}}-e^{-\frac{z}{2}})\,\overset{\sin\theta}{\cancel{\cos\theta}} \\ 1 \end{pmatrix} \qquad T_\theta=\begin{pmatrix} -(e^{\frac{z}{2}}+e^{-\frac{z}{2}})\sin\theta \\ (e^{\frac{z}{2}}+e^{-\frac{z}{2}})\cos\theta \\ 0 \end{pmatrix}$$

$$T_z\times T_\theta=(e^{\frac{z}{2}}+e^{-\frac{z}{2}})\begin{pmatrix} -\cos\theta \\ -\sin\theta \\ \frac{1}{2}(e^{\frac{z}{2}}-e^{-\frac{z}{2}}) \end{pmatrix}$$

$$\|T_z\times T_\theta\|=(e^{\frac{z}{2}}+e^{-\frac{z}{2}})\sqrt{(-\cos\theta)^2+(-\sin\theta)^2+\frac{1}{4}(e^{\frac{z}{2}}-e^{-\frac{z}{2}})^2}$$

$$=(e^{\frac{z}{2}}+e^{-\frac{z}{2}})\sqrt{1+\frac{1}{4}(e^{z}-2+e^{-z})}$$

$$=(e^{\frac{z}{2}}+e^{-\frac{z}{2}})\sqrt{\frac{1}{4}(e^{z}+2+e^{-z})}$$

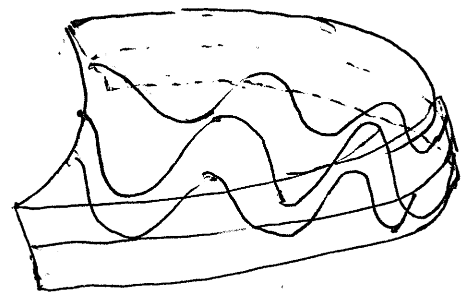
$$=\frac{1}{2}(e^{\frac{z}{2}}+e^{-\frac{z}{2}})^2=\frac{1}{2}(e^{z}+2+e^{-z})$$

- Evaluate the double integral

$$\text{Area}(C)=\int_{-5}^{5}\int_{0}^{2\pi}\frac{1}{2}(e^{z}+2+e^{-z})\,d\theta\,dz$$

$$=\pi\,[e^{z}+2z-e^{-z}]_{z=-5}^{5}=\boxed{2(e^{5}+10-e^{-5})\pi}.$$

3. $\Phi: D \to \mathbb{R}^3$, $D = [-1,1] \times [0, 6\pi]$



$\Phi(u,v) = (x,y,z) = (u, v(1-u^2), \sin v)$

$F(x,y,z) = (x, x^2, z)$

$\vec{T}_u = (1, -2uv, 0)$

$\vec{T}_v = (0, 1-u^2, \cos v)$

$\vec{T}_u \times \vec{T}_v = (-2uv\cos v, -\cos v, 1-u^2)$

$\vec{F}(\Phi(u,v)) = (u, u^2, \sin v)$

$\vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v) = -2u^2 v\cos v - u^2\cos v + (1-u^2)\sin v$

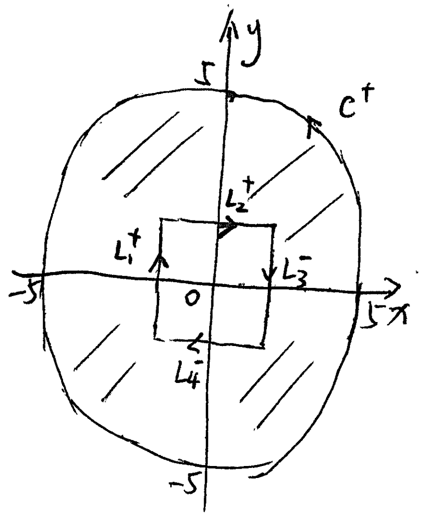
$$\iint_C \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)$$

$$= \int_{-1}^{1} \int_0^{6\pi} (-2u^2 v\cos v - u^2\cos v + (1-u^2)\sin v)\, dv\, du$$

$$= \int_{-1}^{1} 2u^2(v\sin v + \cos v) - u^2 \sin v - (1-u^2)\cos v \Big|_{v=0}^{6\pi} du$$

$$= \int_{-1}^{1} 0\, du = 0.$$

4. The Wuzhu looks like



$$\vec{B}(x,y) = \left(\frac{y}{xy+16}, \frac{x}{xy+16}\right) = (B_1(x,y), B_2(x,y))$$

$\iint_W \nabla\times\vec{B}\cdot\vec{k}\,dA$ will be complicated to evaluate.

Try Green's theorem

$$\partial W = C^+ + L_1^+ + L_2^- + L_3^- + L_4^+$$

$$\iint_W \nabla\times\vec{B}\cdot\vec{k}\,dA = \int_{\partial W}\vec{B}\cdot d\vec{s}$$

1) Parametrize C^+ by $\vec{c}:[0,2\pi]\to\mathbb{R}^2$

$$c(t) = (5\cos t, 5\sin t)$$

$$\int_{C^+}\vec{B}\cdot d\vec{s} = \int_{C^+}\left(\frac{y}{xy+16}, \frac{x}{xy+16}\right)\cdot (dx, dy)$$

$$= \int_0^{2\pi}\left(\frac{5\sin t}{25\sin t\cos t+16}, \frac{5\cos t}{25\sin t\cos t+16}\right)(-5\sin t, 5\cos t)\,dt$$

$$= \int_0^{2\pi}\frac{25(\cos^2 t-\sin^2 t)}{25\sin t\cos t+16}\,dt$$

$u = 25\cos t\sin t$
$= \frac{25}{2}\sin 2t$
$du = 25\cos 2t\,dt$

$$= \int_0^{2\pi}\frac{25\cos 2t}{\frac{25}{2}\sin 2t+16}\,dt$$

$$= \int_{t=0}^{2\pi}\frac{du}{u+16} = \ln|25\cos t\sin t+16|\Big|_{t=0}^{2\pi}$$

$$= 0$$

Or you can use

$$\int_{C^+} \vec{B}\cdot d\vec{s} = \int_{C^+} \frac{y\,dx + x\,dy}{xy+16} = \int_{C^+} \frac{d(xy)}{xy+16}$$

$$= \int_0^{2\pi} \frac{(x(t)\,y(t))'}{x(t)\,y(t)+16}\,dt = \ln\left| x(t)\,y(t) + 16 \right| \Big|_{t=0}^{2\pi}$$

$$= \ln\left| 25\cos t\sin t + 16 \right| \Big|_{t=0}^{2\pi} = 0$$

$L_1^+ = (-2,t),\ t\in[-2,2]$

$$\int_{L_1^+} \vec{B}\cdot d\vec{s} = \int_{-2}^{2} B_2(-2,t)\,dt = \int_{-2}^{2} \frac{-2}{-2t+16}\,dt$$

$$= \int_{-2}^{2} \frac{1}{t-8}\,dt = \ln|t-8| \Big|_{t=-2}^{2} = \ln 6 - \ln 10$$

$L_2^+ = (t,2),\ t\in[-2,2]$

$$\int_{L_2^+} \vec{B}\cdot d\vec{s} = \int_{-2}^{2} B_1(t,2)\,dt = \int_{-2}^{2} \frac{2}{2t+16}\,dt$$

$$= \int_{-2}^{2} \frac{1}{t+8}\,dt = \ln|t+8| \Big|_{t=-2}^{2} = \ln 10 - \ln 6$$

$L_3^+ = (2,t),\ t\in[-2,2]$

$$\int_{L_3^+} \vec{B}\cdot d\vec{s} = \int_{-2}^{2} B_2(2,t)\,dt = \int_{-2}^{2} \frac{2}{2t+16} = \ln 10 - \ln 6$$

$L_4^+ = (t,-2),\ t\in[-2,2]$

$$\int_{L_4^+} \vec{B}\cdot d\vec{s} = \int_{-2}^{2} B_1(t,-2)\,dt = \int_{-2}^{2} \frac{-2}{-2t+16} = \ln 6 - \ln 10$$

Hence $\iint_W \nabla\times\vec{B}\cdot\vec{k}\,dA = \int_{\partial W} \vec{B}\cdot d\vec{s} = \int_{C^+} \vec{B}\cdot d\vec{s} + \int_{L_1^+} \vec{B}\cdot d\vec{s} + \int_{L_2^+} \cdots + \int_{L_3^-} + \int_{L_4^-} \cdots$

$$= \int_{C^+} \vec{B}\cdot d\vec{s} + \int_{L_1^+} \cdots + \int_{L_2^+} \cdots - \int_{L_3} \cdots - \int_{L_4} \cdots$$

$$= 0 + (\ln 6 - \ln 10) + (\ln 10 - \ln 6) - (\ln 10 - \ln 6) - (\ln 6 - \ln 10) = 0.$$