Math 240B: Real Analysis, Winter 2016

Solution to Selected Problems of Homework 2

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Solution to Problem 2. **(1)** Let $f(x) = \lim_{n\to\infty} f_n(x)$ for any $x \in X$ be the pointwise limit, then $f\colon X \to \mathbb{C}$. By uniform Cauchy-ness, for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for ant $n, m \geq N$ and any $x \in X$, $\left|f_n(x) - f_m(x)\right| < \varepsilon$. Fix an $n \geq N$,

$$\left|f_n(x) - f(x)\right| \leq \left|f_n(x) - f_m(x)\right| + \left|f_m(x) - f(x)\right|.$$

Let $m \to \infty$ we get $\left|f_n(x) - f(x)\right| < \epsilon$, which proves $\left\|f_n - f\right\|_u \to 0$. Also $\left\|f\right\|_u \leq \left\|f_n\right\|_u + \varepsilon < +\infty$.

(2) For any $\varepsilon > 0$, for a fixed $n \in \mathbb{N}$ large enough that $\left\|f - f_n\right\|_u < \varepsilon/3$, let $\delta > 0$ that $\left|y - x\right| < \delta \implies \left|f_n(x) - f_n(y)\right| < \varepsilon/3$, which implies

$$\left|f(x) - f(y)\right| \leq \left|f(x) - f_n(x)\right| + \left|f_n(x) - f_n(y)\right| + \left|f_n(y) - f(y)\right| < \varepsilon.$$

So f is continuous. □

Solution to Problem 3. **(1)** Denote

$$A := \{\, x \in X \mid f(x) = g(x) \,\}.$$

For any $x \in A^{\complement}$, there are open neighborhoods $U \ni f(x)$ and $V \ni g(x)$ such that $U \cap V \neq \emptyset$ since Y is Hausdorff. So $G := f^{-1}(U) \cap g^{-1}(V)$ is an open neighborhood of $x \in X$ that any $G \cap A = \emptyset$, so A is closed.

(2) If A is dense and closed, then $A = \overline{A} = X$. □

Solution to Problem 4. Suppose there are $x \neq y \in X$ such that for any $f \in \mathcal{F}$, $f(x) = f(y)$. So no set of the form $f^{-1}(U)$, where $U \subset \mathbb{R}$ is an open subset, separates x and y. Since $\mathcal{T}$ is generated by $f^{-1}(U)$, no open set in $\mathcal{T}$ separates x and y, so $\mathcal{T}$ is not Hausdorff.

Suppose for any $x \neq y \in X$ there is an $f \in \mathcal{F}$ that $f(x) \neq f(y)$, then let $U \ni f(x)$ and $V \ni f(y)$ be disjoint open subsets of $\mathbb{R}$. Then $f^{-1}(U) \ni x$ and $g^{-1}(V) \ni y$ are disjoint opens neighborhoods of x and y. So $\mathcal{T}$ is Hausdorff. □