Solution to Problem 2. (1) Let \( f(x) = \lim_{n \to \infty} f_n(x) \) for any \( x \in X \) be the pointwise limit, then \( f: X \to \mathbb{C} \). By uniform Cauchy-ness, for any \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that for ant \( n, m \geq N \) and any \( x \in X \), \( |f_n(x) - f_m(x)| < \varepsilon \). Fix an \( n \geq N \),
\[
|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|.
\]
Let \( m \to \infty \) we get \( |f_n(x) - f(x)| < \varepsilon \), which proves \( \|f_n - f\|_u \to 0 \). Also \( \|f_n - f\|_u < \varepsilon < +\infty \).

(2) For any \( \varepsilon > 0 \), for a fixed \( n \in \mathbb{N} \) large enough that \( \|f - f_n\|_u < \varepsilon / 3 \), let \( \delta > 0 \) that \(|y - x| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon / 3\), which implies
\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.
\]
So \( f \) is continuous. \( \square \)

Solution to Problem 3. (1) Denote
\[
A := \{ x \in X \mid f(x) = g(x) \}.
\]
For any \( x \in A^C \), there are open neighborhoods \( U \ni f(x) \) and \( V \ni g(x) \) such that \( U \cap V \neq \emptyset \) since \( Y \) is Hausdorff. So \( G := f^{-1}(U) \cap g^{-1}(V) \) is an open neighborhood of \( x \in X \) that any \( G \cap A = \emptyset \), so \( A \) is closed.

(2) If \( A \) is dense and closed, then \( A = \overline{A} = X \). \( \square \)

Solution to Problem 4. Suppose there are \( x \neq y \in X \) such that for any \( f \in \mathcal{T} \), \( f(x) = f(y) \). So no set of the form \( f^{-1}(U) \), where \( U \subset \mathbb{R} \) is an open subset, separates \( x \) and \( y \). Since \( \mathcal{T} \) is generated by \( f^{-1}(U) \), no open set in \( \mathcal{T} \) separates \( x \) and \( y \), so \( \mathcal{T} \) is not Hausdorff.

Suppose for any \( x \neq y \in X \) there is an \( f \in \mathcal{T} \) that \( f(x) \neq f(y) \), then let \( U \ni f(x) \) and \( V \ni f(y) \) be disjoint open subsets of \( \mathbb{R} \). Then \( f^{-1}(U) \ni x \) and \( g^{-1}(V) \ni y \) are disjoint opens neighborhoods of \( x \) and \( y \). So \( \mathcal{T} \) is Hausdorff. \( \square \)