Math 240B: Real Analysis, Winter 2016

Solution to Homework 4

Xiudi Tang
University of California, San Diego

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Solution to Problem 1. Without loss of generality we assume $\mathcal{F}$ is compact. Denote $X = [0,1]$. Since X is compact, $C(X) = BC(X)$.

We use the following Tube Lemma: Let X and Y be topological spaces with Y compact. If U is a neigborhood of $\{x\} \times Y$ in the product space $X \times Y$, then there exists a neighborhood V of x in X such that $V \times Y \subset U$.

For any $x \in X, \varepsilon > 0$, define

$$U_{x,\varepsilon} := \left\{ (y, f) \in X \times \mathcal{F} \;\middle|\; \left|f(y) - f(x)\right| < \varepsilon \right\} \supset \{x\} \times \mathcal{F}.$$

If $(y, f) \in U_{x,\varepsilon}$ then

$$\left|f(z) - f(y)\right| < \frac{\varepsilon}{3} \text{ and } \left\|g - f\right\|_u < \frac{\varepsilon}{3} \implies (z, g) \in U_{x,\varepsilon}.$$

So $U_{x,\varepsilon}$ is open. For any $x \in X, \varepsilon > 0$, by Tube Lemma, there exists a neighborhood $V_{x,\varepsilon}$ of x in X such that for any $y \in V_{x,\varepsilon}$ and any $f \in \mathcal{F}$, $\left|f(y) - f(x)\right| < \varepsilon$. Hence $\mathcal{F}$ is equicontinuous.

Since $\|\cdot\|_u : \mathcal{F} \to \mathbb{C}$ is a continuous map and $\mathcal{F}$ is compact, $\max_{f \in \mathcal{F}} \left\|f\right\|_u$ exist and is finite, so $\mathcal{F}$ is uniformly bounded. □

Solution to Problem 2. Denote $X = [0,1]$. Fix $x \in X, \varepsilon > 0$. Since X is compact, let $\delta > 0$ be such that for any $z \in X, |z - x| < \delta$, $\left|K(z, y) - K(x, y)\right| < \varepsilon$, then we have

$$\left|Tf(z) - Tf(x)\right| \le \int_0^1 \left|K(z, y) - K(x, y)\right| \left|f(y)\right| dy \le \varepsilon \left\|f\right\|_u.$$

So $Tf \in C(X)$.

Let $\mathcal{F} = \left\{ Tf \;\middle|\; \left\|f\right\|_u \le 1 \right\}$. If $f \in \mathcal{F}$, for any $z \in X, |z - x| < \delta$, $\left|K(z, y) - K(x, y)\right| < \varepsilon \implies \left|Tf(z) - Tf(x)\right| \le \varepsilon$, so $\mathcal{F}$ is equicontinuous. Since

$$\left|Tf(x)\right| \le \int_0^1 \left|K(x, y)\right| \left|f(y)\right| dy \le \|K\|_u \left\|f\right\|_u \le \|K\|_u,$$

$\mathcal{F}$ is pointwise bounded. Since X is compact Hausdorff, by Arcelà-Ascoli Theorem, $\mathcal{F}$ is precompact. □

Solution to Problem 3. Let $\mathcal{F} = \left\{ f \in C(X) \,\middle|\, \|f\|_u \leq 1, N_\alpha(f) \leq 1 \right\}$. So $\mathcal{F}$ is pointwise bounded. Since for any $f \in \mathcal{F}$ and $x, y \in X$,

$$\left|f(x) - f(y)\right| \leq N_\alpha(f)\rho(x,y)^\alpha \leq \rho(x,y)^\alpha,$$

$\mathcal{F}$ is uniformly equicontinuous. By Arcelà-Ascoli Theorem, $\mathcal{F}$ is precompact.

Note that $\|\cdot\|_u : C(X) \to \mathbb{C}$ is continuous, and $N_\alpha \colon C(X) \to \mathbb{R}$ is lower semi-continuous since for fixed $x \neq y \in X$, the map

$$\begin{aligned} C(X) &\to \mathbb{R} \\ f &\mapsto \left|f(x) - f(y)\right| \rho(x,y)^{-\alpha} \end{aligned}$$

is continuous. We have $\mathcal{F}$ is closed, so it is compact. □

Solution to Problem 4. Let $\mathcal{A}$ be the complex algebra $\mathcal{C}[C(X), C(Y)]$. Then $\mathcal{A}$ is closed under complex conjugation since complex conjugation is preserved under addition and multiplication. For $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$, assuming without loss of generality $x_1 \neq x_2$, let $f \in C(X)$ that separates x_1 and x_2, then $f \otimes 1_Y \in \mathcal{A}$ separates (x_1, y_1) and (x_2, y_2). By Complex Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X \times Y)$. □