Solution to Problem 1. Without loss of generality we assume $\mathcal{F}$ is compact. Denote $X = [0, 1]$. Since $X$ is compact, $C(X) = BC(X)$.

We use the following Tube Lemma: Let $X$ and $Y$ be topological spaces with $Y$ compact. If $U$ is a neighborhood of $\{x\} \times Y$ in the product space $X \times Y$, then there exists a neighborhood $V$ of $x$ in $X$ such that $V \times Y \subset U$.

For any $x \in X, \epsilon > 0$, define

$$U_{x, \epsilon} := \{(y, f) \in X \times \mathcal{F} \mid \|f(y) - f(x)\| < \epsilon\} \supset \{x\} \times \mathcal{F}.$$ 

If $(y, f) \in U_{x, \epsilon}$ then

$$|f(z) - f(y)| < \frac{\epsilon}{3} \text{ and } \|g - f\|_u < \frac{\epsilon}{3} \implies (z, g) \in U_{x, \epsilon}.$$ 

So $U_{x, \epsilon}$ is open. For any $x \in X, \epsilon > 0$, by Tube Lemma, there exists a neighborhood $V_{x, \epsilon}$ of $x$ in $X$ such that for any $y \in V_{x, \epsilon}$ and any $f \in \mathcal{F}$, $|f(y) - f(x)| < \epsilon$. Hence $\mathcal{F}$ is equicontinuous.

Since $\|\cdot\|_u : \mathcal{F} \to \mathbb{C}$ is a continuous map and $\mathcal{F}$ is compact, $\max_{f \in \mathcal{F}} \|f\|_u$ exist and is finite, so $\mathcal{F}$ is uniformly bounded. \qed

Solution to Problem 2. Denote $X = [0, 1]$. Fix $x \in X, \epsilon > 0$. Since $X$ is compact, let $\delta > 0$ be such that for any $z \in X, |z - x| < \delta$, $|K(z, y) - K(x, y)| < \epsilon$, then we have

$$|Tf(z) - Tf(x)| \leq \int_0^1 |K(z, y) - K(x, y)| \|f(y)\| \, dy \leq \epsilon \|f\|_u.$$ 

So $Tf \in C(X)$.

Let $\mathcal{F} = \{Tf \mid \|f\|_u \leq 1\}$. If $f \in \mathcal{F}$, for any $z \in X, |z - x| < \delta$, $|K(z, y) - K(x, y)| < \epsilon \implies |Tf(z) - Tf(x)| \leq \epsilon$, so $\mathcal{F}$ is equicontinuous. Since

$$|Tf(x)| \leq \int_0^1 |K(x, y)| \|f(y)\| \, dy \leq \|K\|_u \|f\|_u \leq \|K\|_u,$$

$\mathcal{F}$ is pointwise bounded. Since $X$ is compact Hausdorff, by Arzelà-Ascoli Theorem, $\mathcal{F}$ is precompact. \qed
**Solution to Problem 3.** Let \( \mathcal{F} = \{ f \in C(X) \mid \|f\|_u \leq 1, N_\alpha(f) \leq 1 \} \). So \( \mathcal{F} \) is pointwise bounded. Since for any \( f \in \mathcal{F} \) and \( x, y \in X \),

\[
|f(x) - f(y)| \leq N_\alpha(f) \rho(x, y)\alpha \leq \rho(x, y)\alpha,
\]

\( \mathcal{F} \) is uniformly equicontinuous. By Arzelà-Ascoli Theorem, \( \mathcal{F} \) is precompact.

Note that \( \|\cdot\|_u : C(X) \to \mathbb{C} \) is continuous, and \( N_\alpha : C(X) \to \mathbb{R} \) is lower semi-continuous since for fixed \( x \neq y \in X \), the map

\[
C(X) \to \mathbb{R} \\
f \mapsto |f(x) - f(y)| \rho(x, y)^{-\alpha}
\]

is continuous. We have \( \mathcal{F} \) is closed, so it is compact. \( \square \)

**Solution to Problem 4.** Let \( \mathcal{A} \) be the complex algebra \( C[C(X), C(Y)] \). Then \( \mathcal{A} \) is closed under complex conjugation since complex conjugation is preserved under addition and multiplication. For \( (x_1, y_1) \neq (x_2, y_2) \in X \times Y \), assuming without loss of generality \( x_1 \neq x_2 \), let \( f \in C(X) \) that separates \( x_1 \) and \( x_2 \), then \( f \otimes 1_Y \in \mathcal{A} \) separates \( (x_1, y_1) \) and \( (x_2, y_2) \). By Complex Stone-Weierstrass Theorem, \( \mathcal{A} \) is dense in \( C(X \times Y) \). \( \square \)