Math 194 Homework 7

1. Recall that for $0 \leq t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the $\sigma$-algebra $\mathcal{F}_t = \sigma(B_t)$. Show that for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}_t$.

2. Let $W(t)$, $0 \leq t \leq T$ be a Brownian motion, and let $\mathcal{F}_t$ be the associated filtration. Let $\Delta(t)$ be a piecewise constant functions on $[0, T]$, i.e., there is a partition $[0, T] = \bigcup_{j=0}^{n-1} [t_j, t_{j+1})$ such that for every $j$, $\Delta(t) = c_j$ for all $t \in [t_j, t_{j+1})$ and a constant $c_j$. For $t \in [t_j, t_{j+1}]$, define the stochastic process

$$I(t) = \sum_{i=0}^{j-1} \Delta(t_i)[W(t_{i+1}) - W(t_i)] + \Delta(t_j)[W(t) - W(t_j)]$$

. Show that (a) for any $0 \leq s < t \leq T$, the increment $I(t) - I(s)$ is independent of $B_s$; (b) $I(t)$ is a martingale.

3. Find the distribution of $\int_0^1 t^2 B_t dt$.

4. What can you say about $\int_0^1 B_t^2 dt$? (At the least, calculate the mean.)