The Existence and Uniqueness Theorem (of the solution a first order linear equation initial value problem)

Does an initial value problem always a solution? How many solutions are there? The following theorem states a precise condition under which exactly one solution would always exist for a given initial value problem.

**Theorem:** If the functions $p$ and $g$ are continuous on the interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each $t$ in $I$, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where $y_0$ is an arbitrary prescribed initial value.

That is, the theorem guarantees that the given initial value problem will always have (existence of) exactly one (uniqueness) solution, on any interval containing $t_0$ as long as both $p(t)$ and $g(t)$ are continuous on the same interval. The largest of such intervals is called the interval of validity of the given initial value problem. In other words, the interval of validity is the largest interval such that (1) it contains $t_0$, and (2) it does not contain any discontinuity of $p(t)$ nor $g(t)$. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of either $p(t)$ or $g(t)$.

Note that, unless $t_0$ is actually a discontinuity of either $p(t)$ or $g(t)$, there always exists a non-empty interval of validity. If, however, $t_0$ is indeed a discontinuity of either $p(t)$ or $g(t)$, then the interval of validity will be empty. Clearly, in such a case the conditions that the interval must contain $t_0$ and that it must not contain a discontinuity of $p(t)$ or $g(t)$ will be contradicting.
If so, such an initial value problem is not guaranteed to have a unique solution at all.

**Example:** Consider the initial value problem solved earlier
\[ \cos(t)y' - \sin(t)y = 3t\cos(t), \quad y(2\pi) = 0. \]

The standard form of the equation is
\[ y' - \tan(t)y = 3t \]
with \( p(t) = -\tan(t) \) and \( g(t) = 3t \). While \( g(t) \) is always continuous, \( p(t) \) has discontinuities at \( t = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \pm7\pi/2, \ldots \) According to the Existence and Uniqueness Theorem, therefore, a continuous and differentiable solution of this initial value problem is guaranteed to exist uniquely on any interval containing \( t_0 = 2\pi \) but not containing any of the discontinuities. The largest such intervals is \((3\pi/2, 5\pi/2)\). It is the interval of validity of this problem. Indeed, the actual solution \( y(t) = 3t\tan(t) + 3 - 3\sec(t) \) is defined everywhere within this interval, but not at either of its endpoints.
**How to find the interval of validity**

For an initial value problem of a first order linear equation, the interval of validity, if exists, can be found using this following simple procedure.

Given: \( y' + p(t)y = g(t), \quad y(t_0) = y_0. \)

1. Draw the number line (which is the \( t \)-axis).

2. Find all the discontinuities of \( p(t) \), and the discontinuities of \( g(t) \). Mark them off on the number line.

3. Locate on the number line the initial time \( t_0 \). Look for the longest interval that contains \( t_0 \), but contains no discontinuities.

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**Step 1: Draw the \( t \)-axis.**

![Number Line - Step 1](image1)

**Step 2: Mark off the discontinuities.**

![Number Line - Step 2](image2)

**Step 3: Locate \( t_0 \) and determine the interval of validity.**

![Number Line - Step 3](image3)
**Example:** Consider the initial value problems

(a) \[(t^2 - 81)y' + 5e^{3t}y = \sin(t), \quad y(1) = 10\pi\]

(b) \[(t^2 - 81)y' + 5e^{3t}y = \sin(t), \quad y(10\pi) = 1\]

The equation is first order linear, so the theorem applies. The standard form of the equation is

\[
y' + \frac{5e^{3t}}{t^2 - 81}y = \frac{\sin(t)}{t^2 - 81}
\]

with \(p(t) = \frac{5e^{3t}}{t^2 - 81}\) and \(g(t) = \frac{\sin(t)}{t^2 - 81}\). Both have discontinuities at \(t = \pm 9\).

Hence, any interval such that a solution is guaranteed to exist uniquely must contain the initial time \(t_0\) but not contain either of the points 9 and −9.

In (a), \(t_0 = 1\), so the interval contains 1 but not \(\pm 9\). The largest such interval is \((-9, 9)\).

In (b), \(t_0 = 10\pi\), so the interval contains \(10\pi\) but neither of \(\pm 9\). The largest such interval is \((9, \infty)\).

Remember that the value of \(y_0\) does not matter at all; \(t_0\) alone determines the interval.

Suppose the initial condition is \(y(-100) = 5\) instead. Then the largest interval on which the initial value problem’s solution is guaranteed to exist uniquely will be \((-\infty, -9)\).

Lastly, suppose the initial condition is \(y(-9) = 88\). Then we would not be assured of a unique solution at all. Since \(t = -9\) is both \(t_0\) and a discontinuity of \(p(t)\) and \(g(t)\). The interval of validity would be, therefore, empty.
Depending on the problem, the interval of validity, if exists, could be as large as the entire real line, or arbitrarily small in length. The following example is an initial value problem that has a very short interval of validity for its unique solution.

**Example:** Consider the initial value problems

\[(t^2 - 10^{-2000000})y' + ty = 0, \quad y(0) = \alpha.\]

With the standard form

\[y' - \frac{t}{t^2 - 10^{-2000000}} y = 0,\]

the discontinuities (of \(p(t)\)) are \(t = \pm 10^{-1000000}\). The initial time is \(t_0 = 0\).

Therefore, the interval of validity for its solution is the interval \((-10^{-1000000}, 10^{-1000000})\), an interval of length \(2\times10^{-1000000}\) units!

However, the important thing is that somewhere on the \(t\)-axis a unique solution to this initial value problem exists. Different initial value \(\alpha\) will give different particular solution. But the solution will each uniquely exist, at a minimum, on the interval \((-10^{-1000000}, 10^{-1000000})\).

Again, according to the theorem, the only time that a unique solution is not guaranteed to exist anywhere is whenever the initial time \(t_0\) just happens to be a discontinuity of either \(p(t)\) or \(g(t)\).

Now suppose the initial condition is \(y(0) = 0\). It should be fairly easy to see that the constant zero function \(y(t) = 0\) is a solution of the initial value problem. It is of course the unique solution of this initial value problem. Notice that this solution exists for all values of \(t\), not just inside the interval \((-10^{-1000000}, 10^{-1000000})\). It exists even at discontinuities of \(p(t)\). This illustrates that, while outside of the interval of validity there is no guarantee that a solution would exist or be unique, the theorem nevertheless does not prevent a solution to exist, even uniquely, where the condition required by the theorem is not met.
Nonlinear Equations: Existence and Uniqueness of Solutions

A theorem analogous to the previous exists for general first order ODEs.

**Theorem:** Let the function $f$ and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$, containing the point $(t_0, y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

This is a more general theorem than the previous that applies to all first order ODEs. It is also less precise. It does not specify a precise region that a given initial value problem would have a solution or that a solution, when it exists, is unique. Rather, it states a region that somewhere within there has to be part of it in which a unique solution of the initial value problem will exist. (It does not preclude that a second solution exists outside of it.)

The bottom line is that a nonlinear equation might have multiple solutions corresponding to the same initial condition. On the other hand it is also possible that it might not have a solution defined on parts of the region where $f$ and $\frac{\partial f}{\partial y}$ are both continuous.

**Example:** Consider the (nonlinear) initial value problem

$$y' = t^2 y^{1/2}, \quad y(0) = 0.$$

When $t = 0$, $\frac{\partial f}{\partial y}$ is not continuous. Therefore, it would not necessarily have a unique solution. Indeed, both $y = \frac{t^6}{36}$ and $y = 0$ are functions that satisfy the problem. (Verify this fact!)
Exercises A-1.2:

1 – 4 Find the general solution of each equation below.
1. \( y' - t^2 y = 4t^2 \)
2. \( y' + 10y = t^2 \)
3. \( \frac{1}{t^2} y' - e^t y = 0 \)
4. \( y' - y = 2e^t \)

5 – 15 Solve each initial value problem. What is the largest interval in which a unique solution is guaranteed to exist?
5. \( y' + 2y = te^{-t}, \quad y(0) = 2 \)
6. \( y' - 11y = 4e^{6t}, \quad y(0) = 9 \)
7. \( ty' - y = t^2 + t, \quad y(1) = 5 \)
8. \( (t^2 + 1)y' - 2ty = t^3 + t, \quad y(0) = -4 \)
9. \( y' + (2t - 6t^2)y = 0, \quad y(0) = -8 \)
10. \( t^2y' + 4ty = \frac{2}{t}, \quad y(-2) = 0 \)
11. \( (t^2 - 49)y' + 4ty = 4t, \quad y(0) = 1/7 \)
12. \( y' - y = t^2 + t, \quad y(0) = 3 \)
13. \( y' + y = e^t, \quad y(0) = 1 \)
14. \( ty' + 4y = 4, \quad y(-2) = 6 \)
15. \( \tan(t)y' - \sec(t)\tan^2(t)y = 0, \quad y(0) = \pi \)
16 – 19 Without solving the initial value problem, what is the largest interval in which a unique solution is guaranteed to exist for each initial condition?

(a) \( y(\pi) = 7 \),
(b) \( y(1) = -9 \),
(c) \( y(-4) = e \).

16. \((t + 5)y' + \frac{(t-8)(t-1)}{t-3}y = \frac{t}{(t-6)(t+1)}\)

17. \(t^2y' + \frac{t-2}{t+3}y = \sec(t/3)\)

18. \((t^2 + 4t - 5)y' + \tan(2t)y = t^2 - 16\)

19. \((4 - t^2)y' + \ln(6 - t)y = e^{-t}\)

20. Find the general solution of \( t^2y' + 2ty = 2 \). Then show that both the initial conditions \( y(1) = 1 \) and \( y(-1) = -3 \) result in an identical particular solution. Does this fact violate the Existence and Uniqueness Theorem?
Answers A-1.2:

1. \( y = -4 + Ce^{t^3/3} \)

2. \( y = \frac{t^2}{10} - \frac{t}{50} + \frac{1}{500} + Ce^{-10t} \)

3. \( y = C \exp\left(\frac{1}{3}e^t\right) \)

4. \( y = 2te^t + Ce^t \)

5. \( y = te^{-t} - e^{-t} + 3e^{-2t}, \quad (-\infty, \infty) \)

6. \( y = \frac{49}{5}e^{11t} - \frac{4}{5}e^{6t}, \quad (-\infty, \infty) \)

7. \( y = t^2 + t\ln t + 4t, \quad (0, \infty) \)

8. \( y = (t^2 + 1)(\ln(\sqrt{t^2 + 1} - 4), \quad (-\infty, \infty) \)

9. \( y = -8\exp(2t^3 - t^2), \quad (-\infty, \infty) \)

10. \( y = \frac{1}{t^2} - \frac{4}{t^4}, \quad (-\infty, 0) \)

11. \( y = \frac{t^4 - 98t^2 + 343}{(t^2 - 49)^2}, \quad (-7, 7) \)

12. \( y = 6e^t - t^2 - 3t - 3, \quad (-\infty, \infty) \)

13. \( y = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \quad (-\infty, \infty) \)

14. \( y = 1 + 80t^{-4}, \quad (-\infty, 0) \).

15. \( y = \frac{\pi}{e}\sec(t) = \pi e^{\sec(-t)}, \quad (-\pi/2, \pi/2) \)

16. (a) (3, 6); (b) (-1, 3); (c) (-5, -1).

17. (a) (0, 3\pi/2); (b) (0, 3\pi/2); (c) (-3\pi/2, -3).

18. (a) (3\pi/4, 5\pi/4); (b) no such interval exists; (c) (-5, -5\pi/4).

19. (a) (2, 6); (b) (-2, 2); (c) (-\infty, -2).

20. \( y = \frac{2t + C}{t^2} \); they both have \( y = \frac{2t - 1}{t^2} \) as the solution; no, different initial conditions could nevertheless give the same unique solution.