Math 3C Section 1.3

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1 Inequalities of real numbers

If we pick any two real numbers a and b, then we can tell whether $a < b$, $a = b$ or $a > b$. This relation is called an order on real numbers. An inequality expresses such an order between two quantities, like

$$a < b, \quad a + b < c + d, \quad \frac{x-2}{5} < 6.$$

With such an order in our mind, we can divide all real numbers into three classes: positive numbers, 0 and negative numbers. A real line is a good way to illustrate such a division.

With a real line it is easy to understand three basic properties of inequalities of real numbers.

Transitivity If $a < b$ and $b < c$, then $a < c$.
Addition of inequalities If $a < b$ and $c < d$, then $a + c < b + d$.
Multiplication of inequalities Suppose $a < b$, if $c > 0$ then $ac < bc$; if $c < 0$ then $ac > bc$. (*in other words, scaling by $c > 0$ preserves inequality; scaling by $c < 0$ means flipping signs and scaling, it reverses the inequality.*)

A property not geometrically obvious is

Multiplicative inverse of inqualities Suppose $a < b$. If a and b are both positive of negative, then $1/a > 1/b$; if $a < 0 < b$ then $1/a < 1/b$.

Examples. Consider real numbers $1, \sqrt{3}, \sqrt{5}, \sqrt{7}$.
1. list the above four numbers in an ascending order.
2. Use the property of addition of inequalities to show that $\sqrt{5} + 1 < \sqrt{7} + \sqrt{3}$.
3. Can you use the property of addition of inequalities to compare $\sqrt{5} - 1$ and $\sqrt{7} - \sqrt{3}$? Why or why not?
4. (A bit challenging) Can you use one of the properties above to show that $\sqrt{5} - 1 > \sqrt{7} - \sqrt{3}$?

Solution.
1. Because the square of each positive number can be listed in ascending order

as $1 < 3 < 5 < 7$, we have $1 < \sqrt{3} < \sqrt{5} < \sqrt{7}$.

2. Since $\sqrt{5} < \sqrt{7}$ and $1 < \sqrt{3}$, by addition property we see the conclusion.

3. No, because $\sqrt{5} < \sqrt{7}$ but $-1 > -\sqrt{3}$, also $\sqrt{5} > -\sqrt{3}$ but $-1 < \sqrt{7}$, hence we cannot use addition property.

4. we cannot directly use the addition property to compare these numbers, but we can use the inverse property. We shall use the following formula

$$a^2 - b^2 = (a+b)(a-b) \quad \text{for any real numbers } a \text{ and } b$$

dividing both sides by $(a^2 - b^2)(a-b)$ we get

$$\frac{1}{a-b} = \frac{a+b}{a^2-b^2}.$$

If we take the inverse of $\sqrt{7} - \sqrt{3}$ and $\sqrt{5} - \sqrt{1}$, then we see

$$\frac{1}{\sqrt{7}-\sqrt{3}} = \frac{\sqrt{7}+\sqrt{3}}{(\sqrt{7})^2 - (\sqrt{3})^2} = \frac{\sqrt{7}+\sqrt{3}}{7-3} = \frac{\sqrt{7}+\sqrt{3}}{4}$$
$$\frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+\sqrt{1}}{(\sqrt{5})^2 - (1)^2} = \frac{\sqrt{5}+1}{5-1} = \frac{\sqrt{5}+1}{4}$$

Since the denominator of both fractions are 4, and numerators can be compared using addition property $\sqrt{5} + 1 < \sqrt{7} + \sqrt{3}$, hence

$$\frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4} < \frac{\sqrt{7}+\sqrt{3}}{4} = \frac{1}{\sqrt{7}-\sqrt{3}}$$

hence by the inverse property,

$$\sqrt{7} - \sqrt{3} < \sqrt{5} - 1.$$

2 Set of real numbers, intervals and absolute values

A set of real numbers is a collection of real numbers chosen according to some properties. For example,

$$\{x : x < 0\}$$

is the set of negative numbers, whereas

$$\{x : x \text{ is an integer}\}$$

is the set of integers. In this course we focus on a special type of set: intervals. There are four kinds of intervals:

Closed: $[a, b] := \{x : a \leq x \leq b\}$
Open: $(a, b) := \{x : a < x < b\}$
Left closed, right open: $[a, b) := \{x : a \leq x < b\}$
Left open, right closed: $(a, b] : \{x : a < x \leq b\}$

We extend the notion of interval to the following sets:

$$(a, \infty) := \{x : x > a\}, \quad (-\infty, a) := \{x : x < a\}$$

and

$$[a, \infty) := \{x : x \geq a\}, \quad (-\infty, a] := \{x : x \leq a\}$$

2.1 Union and intersection of intervals

Let A and B be intervals. The union of A and B, denoted by $A \cup B$, is the set consists of elements either in A or B. Meaning: take elements in A and B and put them together with common elements counted once. The intersection of A and B, denoted $A \cap B$ is the set consists of elements simultaneously in A and B.

Using real line we can see clearly how to take union and intersections. For example,

Example. Let $A = (1, 5)$ and $B = (3, 7)$. What is $A \cup B$? And $A \cap B$?

Solution. $A \cup B = (1, 7)$ and $A \cap B = (3, 5)$.

2.2 Absolute Values

For a real number b, define its absolute value to be the distance of b on the real line to the origin 0. Hence it is always nonnegative. we have

$$|b| = \begin{cases} b & b \geq 0 \\ -b & b < 0 \end{cases}$$

Absolute values can be related to intervals in the following expression:

$$\{x : |x - a| < b\} = (-b + a, b + a)$$

It is a good way to denote a "neighborhood" of some real number: the above set contains numbers whose distance to a is less than b.

To deal with problems involving absolute values, the standard method is to split all cases, for example

Example.
1. Express the following sets as an interval:

$$|x - 1| < 3$$

Solution. We can view $x-1$ as a "black box", just use the definition that absolute value of a is the distance from 0 to a, hence the distance from 0 to $x-1$ is less than 3, therefore

$$-3 < x - 1 < 3.$$

hence $-2 < x < 4$ by adding 1 on each term.

2. Find x such that $|x-1| + |x-5| = 9$.

Solution. We shall no more consider $x-1$ or $x-5$ as a "black box" since it does not help. Direct expansion brings uncertainty in the sign of $x-1$ and $x-5$ too. Hence we consider another method that works for such kind of questions consistently: using 1 and 5 as bounds to split all real numbers into intervals. Then we search for x in each of these intervals. Namely we have 3 cases:

Case 1 when $x < 1$, we have $x-1 < 0$, $x-5 < 0$, by the definition of absolute value we have

$$|x-1| + |x-5| = -(x-1) + [-(x-5)] = -2x + 6.$$

Case 2 when $1 \leq x < 5$, we have $x-1 \geq 0$, $x-5 < 0$, by the definition of absolute value we have

$$|x-1| + |x-5| = (x-1) + [-(x-5)] = 4.$$

Case 3 when $x \geq 5$, we have $x-1 > 0$, $x-5 \geq 0$, by the definition of absolute value we have

$$|x-1| + |x-5| = (x-1) + (x-5) = 2x - 6.$$

Hence we have the following

$$|x-1| + |x-5| = \begin{cases} -2x+6 & \text{when } x < 1 \\ 4 & \text{when } 1 \leq x < 5 \\ 2x-6 & \text{when } x \geq 5 \end{cases}$$

Therefore $|x-1| + |x-5| = 9$ has two solutions $x = -3/2$ and $x = 15/2$ belonging to $x < 1$ and $x \geq 5$ respectively. There is no solution x in $[1, 5)$ since $|x-1| + |x-5|$ is identically 4 in that case.

A Further Question How do we apply similar methods to the equation containing 3 or more absolute values like

$$|x-1| + |x-2| + |x-3| = 9$$

Sometimes it is useful to refer to real line: interpreting the absolute value as expressing a distance of some number to another. This helps a lot when arguing the non existence of solutions. For example, does there exist a solution to the following equation?

$$|x-1|+|x-5|=2$$

The answer is no, since the sum of distance from any point x to two distinct points 1 and 5 cannot be smaller than the distance between 1 and 5. (use a real line to see this!) Therefore the left hand side cannot be $2<4$.