Chapter 2.1

1 Coordinate Plane

A plane is an infinitely extended 2-dimensional object, which is a generalization of the 1-dimensional straight line. Analogous the construction of real line, we can associate a pair of real numbers \((x, y)\) to every point on the plane, to get the so-called Coordinate plane. The Cartesian coordinate (due to Descarte) has the following structure:

The \(x\)-axis and \(y\)-axis are perpendicular, and their intersection has coordinates \((0, 0)\), and is called the origin. The points on \(x\)-axis and \(y\)-axis have coordinates \((a, 0)\) and \((0, b)\) respectively. To find the coordinate of a given point, drop two lines, one parallel to \(x\)-axis and the other parallel to \(y\)-axis, then the intersection on \(x\)-axis is the \(x\) coordinate value and the other intersection on \(y\)-axis is the \(y\) coordinate. \(x,y\)-axis divide the plane into four parts, called quadrants. The quadrant in which points have \((+, +)\) sign of their coordinates is called first quadrant, \((-+, +)\) called second quadrant, \((--, -)\) called third quadrant and \((+, -)\) called fourth quadrant.

The significance of the coordinate plane is that it builds a correspondence between algebraic equations and curves on the plane. For example, equation \(y = x\) corresponds to the line on which every point has equal \(x\) and \(y\) coordinates. It is a straight line pass through the origin \((0,0)\) with 45 degree slope.

**Question.** Which curve does the equation \(x^2 + y^2 = 1\) correspond to? Describe it precisely.

**Answer.** After plugging in some special points \((\pm 1, 0), (0, \pm 1)\) and \((\pm \sqrt{1/2}, \pm \sqrt{1/2})\) and plotting them on \(xy\)-plane, we reasonably guess that the curve is a circle. It is actually a circle of radius 1 centered at the origin. Proof will be given later.

2 Distance of Two Points on \(xy\)-plane

One of the basic formulae tells us the distance between points in the plane given their coordinates. If point \(A\) is \((x_1, y_1)\) and point \(B\) is \((x_2, y_2)\), then their
distance is
\[ \text{dist}(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \]
The proof is simple: dropping a vertical line from \( A \) and a horizontal line from \( B \) we denote the intersection by \( C \), then \( C \) has coordinate \((x_1, y_2)\). (The figure is omitted.) \( ABC \) is a right triangle with hypotenuse \( AB \). By Pythagorean Theorem
\[ |AB|^2 = |BC|^2 + |AC|^2, \]
and \( |BC| = |x_1 - x_2|, \) \( |AC| = |y_1 - y_2| \) since they are horizontal and vertical respectively. Hence
\[ \text{dist}(A, B) = |AB| = \sqrt{|AC|^2 + |BC|^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \]

**Example.** What is the distance between \((1, 0)\) and \((-5, 7)\)?

**Solution.** by distance formula we have
\[ d = \sqrt{|1 - (-5)|^2 + (0 - 7)^2} = \sqrt{36 + 49} = \sqrt{85}. \]

Now we can see why \( x^2 + y^2 = 1 \) is a unit circle centered at \((0,0)\). By the distance formula, the distance of an arbitrary point \((x, y)\) to \((0,0)\) is \( \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} \). If \( x, y \) satisfies \( x^2 + y^2 = 1 \), then \( \sqrt{x^2 + y^2} = 1 \). This means that the distance of points on the curve and \((0,0)\) is identically \(1\). That defines exactly a unit circle centered at the origin.

**Question.** Can you describe the curve corresponding to equation
\[ (x - 1)^2 + (y - 3)^2 = 5^2 \]
in a similar manner? (Answer. It is a circle of radius 5 centered at \((1,3)\).)

### 3 Perimeter and Circumference

One of the advantage of setting up coordinate is to use algebraic equations to study geometry. For a plane figure bounded by curves or straight lines, we have the concept of “total length”:

**Perimeter:** usually used for figure with corners, like triangles and half discs

**Circumference:** usually used for figure with smooth boundary, like circles and ellipses.

**Example:** Find the perimeter of the triangle with vertices \((1,0)\), \((0,1)\) and \((0,0)\). (Solution will be posted after the lecture!)

Let’s consider the circle. What is the circumference of a circle of radius \( r \)? We introduce a constant \( \pi \) to derive the formula.
Greek mathematicians proved that the circumference of the circle is proportional to its diameter: the proportional constant is called $\pi$. Hence the formula is

$$l = \pi d = 2\pi r.$$ 

**Example.** What is the perimeter of a half disk of radius 2? (Solution will be posted after the lecture!)