Math 3C Section 2.1

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1 Coordinate Plane

A plane is an infinitely extended 2-dimensional object, which is a generalization of the 1-dimensional straight line. Analogous the construction of real line, we can associate a pair of real numbers (x, y) to every point on the plane, to get the so-called Coordinate plane. The Cartesian coordinate(due to Descarte) has the following structure:

The x-axis and y-axis are perpendicular, and their intersection has coordinates $(0, 0)$, and is called the origin. The points on x-axis and y-axis have coordinates $(a, 0)$ and $(0, b)$ respectively. To find the coordinate of a given point, drop two lines, one parallel to x-axis and the other parallel to y-axis, then the intersection on x-axis is the x coordinate value and the other intersection on y-axis is the y coordinate. x,y-axis divide the plane into four parts, called **quadrants**. The quadrant in which points have $(+, +)$ sign of their coordinates is called **first quadrant**, $(-, +)$ called **second quadrant**, $(-, -)$ called **third quadrant** and $(+, -)$ called **fourth quadrant**.

The significance of the coordinate plane is that it builds a correspondence between algebraic equations and curves on the plane. For example, equation $y = x$ corresponds to the line on which every point has equal x and y coordinates. It is a straight line pass through the origin $(0, 0)$ with 45 degree slope.

Question. Which curve does the equation $x^2 + y^2 = 1$ correspond to? Describe it precisely.
Answer. After pluging in some special points $(\pm 1, 0)$, $(0, \pm 1)$ and $(\pm\sqrt{1/2}, \pm\sqrt{1/2})$ and plotting them on xy-plane, we reasonably guess that the curve is a circle. It is actually a circle of radius 1 centered at the origin. Proof will be given later.

2 Distance of Two Points on xy-plane

One of the basic formulae tells us the distance between points in the plane given their coordinates. If point A is (x_1, y_1) and point B is (x_2, y_2), then their distance is

$$\text{dist}(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The proof is simple: dropping a vertical line from A and a horizontal line from B we denote the intersection by C, then C has coordinate (x_1, y_2). (The figure is omitted.) ABC is a right triangle with hypotenuse AB. By Pythagorean Theorem

$$|AB|^2 = |BC|^2 + |AC|^2,$$

and $|BC| = |x_1 - x_2|$, $|AC| = |y_1 - y_2|$ since they are horizontal and vertical respectively. Hence

$$\text{dist}(A, B) = |AB| = \sqrt{|AC|^2 + |BC|^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example. What is the distance between $(1, 0)$ and $(-5, 7)$?
Solution. by distance formula we have

$$d = \sqrt{[1 - (-5)]^2 + (0 - 7)^2} = \sqrt{36 + 49} = \sqrt{85}.$$

Now we can see why $x^2 + y^2 = 1$ is a unit circle centered at 0. By the distance formula, the distance of an arbitrary point (x, y) to $(0, 0)$ is $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$. If x, y satisfies $x^2 + y^2 = 1$, then $\sqrt{x^2 + y^2} = 1$. This means that the distance of points on the curve and $(0, 0)$ is identically 1. That defines exactly a unit circle centered at the origin.

Question. Can you describe the curve corresponding to equation

$$(x - 1)^2 + (y - 3)^2 = 5^2$$

in a similar manner? (*Answer.* It is a circle of radius 5 centered at $(1, 3)$.)

3 Perimeter and Circumference

One of the advantage of setting up coordinate is to use algebraic equations to study geometry. For a plane figure bounded by curves or straight lines, we have the concept of "total length":

Perimeter: usually used for figure with corners, like triangles and half discs
Circumference: usually used for figure with smooth boundary, like circles and ellipses.

Example: Find the perimeter of the triangle with vertices $(1, 0)$, $(0, 1)$ and $(0, 0)$. (**Solution will be posted after the lecture!**)

Let's consider the circle. What is the circumference of a circle of radius r? We introduce a constant π to derive the formula.

Greek mathematicians proved that the circumference of the circle is proportional to its diameter: the proportional constant is called π. Hence the formula is

$$l = \pi d = 2\pi r.$$

Example. What is the perimeter of a half disk of radius 2? (**Solution will be posted after the lecture!**)