Math 3C Section 2.2

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1 Slope of Lines

To find the equation of straight lines in coordinate plane, we need a parameter to characterize its direction. Suppose you are walking along a line which is neither vertical or horizontal. Then your motion is a combination of horizontal and vertical displacement. During a time interval, you walk from (x_1, y_1) to (x_2, y_2) along this line, then your horizontal and vertical displacement are

$$h = \text{hori. disp} = x_2 - x_1, \quad v = \text{vert. disp} = y_2 - y_1.$$

The characterization of a staight line is that the ratio v/h keeps a constant whichever point you start from or end at, provided that you stay on the line. This is what we call by slope of a line.

Slope $k = v/h$ can be positive, meaning that the line is uphill; or negative, meaning that the line is downhill. $k = 0$ implies vertical displacement $v = 0$ whatever h is. There is also special lines which do not have a slope: the vertical lines. Since on each vertical line, the horizontal displacement $h = 0$, hence $k = v/h = v/0$ does not exist.

Question. For a given line l, how do we practically find its slope?
Solution. Pick any two points on line l, say (x_1, y_1) and (x_2, y_2). Then

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

2 Equation of Lines

There are several forms of equation of a straight line, depending on which information of the line is given initially. We talk about 3 types of equations of a line l:

(1) given slope k and a point (x_1, y_1) that l passes through

$$y - y_1 = k(x - x_1)$$

(2) given slope k and y-intercept b;

$$y = kx + b$$

(3) given two points (x_1, y_1) and (x_2, y_2) on l.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Example 1. If m is an arbitrary real number, in which quadrant(s) **cannot** $(m, m+3)$ be? Likewise, in which quadrant(s) **cannot** $(m-2, 9-3m)$ be?

Solution. At first it seems we have to discuss about each possible cases, for example, If $m < 0$, then $m+3$ can be negative(when $m < -3$) or positive(when $-3 < m < 0$). Hence the point $(m, m+3)$ can be present in second and third quadrant. The other case is when $m > 0$. That implies $m + 3 > 0$, hence $(m, m+3)$ can only be in the first quadrant if $m > 0$. Summing up we see that $(m, m+3)$ can be in the first, second and third quadrant, hence the answer is fourth quadrant.

After studying the equation of straight lines, another(possibly faster) method is available: we consider the collection of all points $(m, m+3)$. This amounts to say we are looking at point with their y-coordinate being greater than their x-coordinate by 3, i.e. $y = x + 3$. This is a straight line with slope 1 and y-intercept 3. After plotting the graph, we see immediately that the line does not pass through fourth quadrant.

For the second point $(m-2, 9-3m)$, it may not be obvious what kind of straight lines this should be. But think about that we do not need to find the equation in order to plot the line. We only need two special points on it. Pick two points which tell you the x and y-intercept. For example, $m = 2$ gives $(0, 3)$ for y-intercept, $m = 3$ gives $(1, 0)$ for x-intercept. Connecting $(0, 3)$ and $(1, 0)$ and extend the line in both direction, so we can see this line does not pass through the third quadrant.

3 Parallel and Perpendicular Lines

Question. If two lines l_1 and l_2 are parallel or perpendicular, what can we say about their equations?
Answer. If we use the second equation of lines, let $y = k_1 x + b_1$ be for l_1, $y = k_2 x + b_2$ be for l_2. For parallel lines we have $k_1 = k_2$. Also note that $b_1 \neq b_2$ since otherwise l_1 and l_2 are the same line. If l_1 and l_2 are perpendicular, then $k_1 \cdot k_2 = -1$, and we do not have restriction on b_1, b_2, as l_1, l_2 can never be the same if they are perpendicular.

Example. Show that the quadrilateral with vertices at $A = (0, 0), B = (2, 1), C = (5, 4), D = (3, 3)$ is a parallelogram.

Solution. We show that the opposite sides are parallel. After plotting the quadrilateral we see we need to show $AB \parallel CD$ and $AD \parallel BC$. The slope of each segment:

$$k_{AB} = \frac{1-0}{2-0}, \quad k_{CD} = \frac{4-3}{5-3}, \quad k_{AD} = \frac{3-0}{3-0}, \quad k_{BC} = \frac{4-1}{5-2}$$

Hence $k_{AB} = k_{CD} = 1/2$ and $k_{AD} = k_{BC} = 1$, therefore we have $AB \parallel CD$ and $AD \parallel BC$.

Example. As another example, we show that the quadrilateral with vertices $A = (-1, 0), B = (3, 1), C = (1, 9), D = (-3, 8)$ is a rectangle.
Solution. In this case we only need to show that each pair of adjacent sides are perpendicular. That is, $AB \perp BC$, $BC \perp CD$, $CD \perp DA$ and $DA \perp AB$. In practice we only need to show three of them, since the sum of all angles are $360°$, the fourth angle is automatically $90°$ if three of them are $90°$ each. We have

$$k_{AB} = \frac{1-0}{3-(-1)}, \quad k_{BC} = \frac{9-1}{1-3}, \quad k_{CD} = \frac{8-9}{-3-1}, \quad k_{DA} = \frac{0-8}{-1-(-3)}$$

Simplifying them we see

$$k_{AB} \cdot k_{BC} = \frac{1}{4} \cdot \frac{8}{-2} = -1$$

and similarly

$$k_{BC} \cdot k_{CD} = k_{CD} \cdot k_{DA} = -1$$

Hence we showed that adjacent sides are perpendicular, therefore $ABCD$ is a rectangle.

4 Midpoints

If we pick two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$ on the plane, then their midpoint C has coordinates as the average of those of A and B, which is

$$C : \Big(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\Big).$$