Math 3C Section 2.3

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10/02/2018

1 Quadratic Formula

1.1 Complete Squares

For a quadratic expression $x^2 + bx + c$, it is sometimes useful to eliminate the middle term bx, and rewrite the whole expression as a complete square plus a constant term. The basic formula for such a technique is the expansion of binomial formula:

$$(x + r)^2 = x^2 + 2rx + r^2.$$

Therefore we have

$$x^2 + 2rx = (x + r)^2 - r^2$$

If we let $r = b/2$, then $b = 2r$ and we have

$$x^2 + bx = x^2 + 2rx = (x + r)^2 - r^2 = \big(x + \frac{b}{2}\big)^2 - \big(\frac{b}{2}\big)^2$$

and finally

$$x^2 + bx + c = \big(x + \frac{b}{2}\big)^2 - \big(\frac{b}{2}\big)^2 + c \tag{1}$$

This formula is particularly useful when we deal with extremal problems, i.e. problems of finding maximum or minimum value of a quadratic expression.

Let's start with the simplest example. Consider x^2, since it is always non-negative (as $(-1)^2 = 1 > 0$, $0^2 = 0$, $1^2 = 1 > 0$), the minimum value of x^2 is 0, which is achieved only when $x = 0$. Likewise, the expression $(x + \frac{b}{2})^2$ is always non-negative, and it achieves minimum 0 only when $x + \frac{b}{2} = 0$, i.e. $x = -\frac{b}{2}$.

Hence by rewriting $x^2 + bx + c$ as

$$(x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c,$$

it is clear that minimum is achieved only when $x + \frac{b}{2} = 0$, since $-(\frac{b}{2})^2 + c$ is a constant not depending on x. We look at the following classical example for an application.

Example Find the point on the line $y = 2x - 1$ that is closest to point $(2, 1)$.

Solution. The problem is asking for the closest point, so first we use the distance formula to find an expression for all possible points on the line in *Step 1. & 2.*

Step 1. Let (x, y) be an arbitrary point on the plane. Then

$$\text{dist}[(2,1),(x,y)] = \sqrt{(x-2)^2 + (y-1)^2}.$$

Step 2. Since we look for the closest point on the line $y = 2x - 1$, we need to plug it into the formula above, to get the distance of $(2, 1)$ and $(x, 2x - 1)$:

$$\text{dist}[(2,1),(x,2x-1)] = \sqrt{(x-2)^2 + (2x-1-1)^2}$$

Step 3. Simplify the above expression:

$$\begin{aligned}\sqrt{(x-2)^2 + (2x-1-1)^2} &= \sqrt{(x-2)^2 + (2x-2)^2} \\ &= \sqrt{(x^2-4x+4) + (4x^2-8x+4)} \\ &= \sqrt{5x^2-12x+8}\end{aligned}$$

Step 4. We need to find out the value of x such that the minimum value of $\sqrt{5x^2 - 12x + 8}$ is achieved. We look for x which makes $5x^2 - 12x + 8$ achieve minimum, and such x also makes $\sqrt{5x^2 - 12x + 8}$ its minimum. Using the complete square technique:

$$\begin{aligned}5x^2 - 12x + 8 &= 5\Big(x^2 - \frac{12}{5}x + \frac{8}{5}\Big) \qquad \text{let } b = -\frac{12}{5} \text{ in the formula (1)} \\ &= 5\Big[(x - \frac{6}{5})^2 - \big(\frac{6}{5}\big)^2 + \frac{8}{5}\Big] \qquad \text{hence } \frac{b}{2} = -\frac{6}{5}\end{aligned}$$

Step 5. We can see that the minimum is achieved when $x - \frac{6}{5} = 0$. Plug $x = \frac{6}{5}$ back into the equation of the line $y = 2x - 1$, the closest point on the line to $(2, 1)$ is $(\frac{6}{5}, 2 \cdot \frac{6}{5} - 1)$, i.e. $(\frac{6}{5}, \frac{7}{5})$.

Excursion to geometry: Since we know that the segment from $(2, 1)$ and perpendicular to the line $y = 2x-1$ has the shortest distance, can you use other knowledge from **Chapter 2** to find out the foot of perpendicularity? This is more direct than the preceding approach. (**Hint**: Find the equation of the line containing that perpendicular segment.)

1.2 Quadratic Equations

A quadratic equation has the following form

$$ax^2 + bx + c = 0.$$

Using the complete square technique, we get the following formula

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Some discussion:
Case 1. When $b^2 - 4ac > 0$ then $\sqrt{b^2 - 4ac} > 0$, hence there are two distinct real roots

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Case 2. When $b^2 - 4ac = 0$, then we have no square root term and only one root(called double root)

$$x_1 = x_2 = -\frac{b}{2a}$$

Case 3. When $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number, hence we do not have real roots.

No examples yet, will be added if a good one is found.