1 Function and Domain

A function is a rule that describes, given a value of variable \( x \), how to associate it with another unique real number \( y \). For example, \( y = x + 1 \). The \( y \) value is also frequently refer to as "function", like in the statement: \( y \) is a function of \( x \). The name of variable and function and is of little importance and contingent on the context. For example, \( y = x + 1 \) and \( z = t + 1 \) shall be considered as the same function, used in different situations: dad(\( y \)) is one year elder than mom(\( x \)), and I(\( z \)) am one year elder than my brother(\( t \)). We choose different name for variable but they expressed the same relation between ages.

The usual notation for function is \( y = f(x) \), where \( f \) alone means the mapping rule. We can also think of \( f \) as a machine which takes imput \( x \) and produces output \( y \). But a machine cannot take all kinds of imput since its capacity is limited. So we have the notion of domain of a function.

**Domain** of a function is the set of real numbers that can be taken as imputs by the function.

There are two cases when a function \( y = f(x) \) is given:
(1) the domain of \( f \) is specified (e.g. \( y = 2x + 3 \), \( x \) in \([-1, 1]\).)
(2) the domain of \( f \) is not specified due to laziness (e.g. \( y = \sqrt{x-2} \).)

For case (1), we just need to keep in mind that the domain is an equally important part as the expression of \( f \). Functions like

\[
\begin{align*}
    f(x) &= |x|, x \text{ in } [-1, 1] \\
    f(x) &= |x|, x \text{ is a real number}
\end{align*}
\]

are completely different functions. For case (2), the principle is: domain is the largest possible set of real numbers \( x \) so that \( f(x) \) can be computed to be a real number. There are 2 frequently tested patterns in exercises: square root and fractional expressions. Let’s look at the following sequence of examples:

**Example 1.** Find the domain of \( f(x) = \sqrt{x-2} \).

**Solution.** Since the square root of a negative number is not a real number, we need \( x - 2 \geq 0 \). Therefore the domain is \( \{ x : x \geq 2 \} \), or just \( [2, \infty) \).
Example 2. Find the domain of \( f(x) = \frac{1}{x - 2} \).

Solution. As \( x - 2 \) appears on the denominator, it can not be 0. So the domain is \( \{ x : x \neq 2 \} \) or \( (-\infty, 2) \cup (2, \infty) \).

Example 3. Find the domain of \( f(x) = \frac{\sqrt{x + 4}}{x - 2} \).

Solution. First we see the square root and fraction patterns in the function \( f \). The final operation is the square root, which means that \( \frac{x + 4}{x - 2} \geq 0 \).

Also we observe that \( x - 2 \neq 0 \) since it is the denominator. Let us deal with the fraction inequality above. Since we do not know whether \( x - 2 \) is positive or negative, we cannot multiply \( x - 2 \) to both sides and keep the \( \geq \) sign (see the multiplicative property of inequalities), therefore we multiply \( (x - 2)^2 \) to both sides. Hence we have

\[
\frac{x + 4}{x - 2} \cdot (x - 2)^2 \geq 0 \cdot (x - 2)^2
\]

\[
\implies (x + 4)(x - 2) \geq 0
\]

To solve the above inequality, we have the following simple method (it is a variation of method of graphs): solve equations \( x + 4 = 0 \) and \( x - 2 = 0 \), get \( x = -4 \) and \( x = 2 \). Find \(-4\) and \(2\) on the real line. Taking \( x \) to be a sufficiently large number, like 100, we have \((100 + 4)(100 - 2) > 0\), therefore we start with a point above the real line on the far right. Then we trace back towards the real line and hit it at \( x = 2 \), entering the lower half of the plane. The curve comes back to the real line at \(-4\) which is another zero point. Since there is no more zero point on the left the curve will keep above the line. Observe that on \([2, \infty)\) and \((\infty, 4] \) the curve is above the line, the solution to the inequality is

\[
x \text{ in } [2, \infty) \cup (-\infty, 4]
\]

Considering that \( x \neq 2 \), the domain of \( f \) is

\[
(2, \infty) \cup (-\infty, 4].
\]

Next we talk about the range of a function, which is the set of values of function when the variable varies over the domain. To find the range of a function \( y = f(x) \) on some domain, it is sometimes tricky when we only know the explicit formula \( f(x) \). A more convenient way is to find the graph of \( f \) and extract information directly from the graph.

2 Graph of Functions

(Please see the handwritten notes for this section.)