Math 3C Section 3.2

Yucheng Tu

10/16/2018

1 Introduction

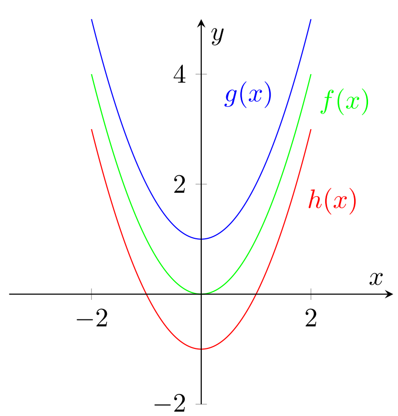
Graph of functions is a superb way of studying the behavior of functions, like its monontonicity in a given interval(increasing or decreasing) and its extremal(maximum and minimum) points. But for a function with complicated expression, like $y = \frac{x^2-2x+4}{x+1}$, it is not easy to sketch its graph. We are going to study the transformation of functions so that we can sketch graphs of lots of functions using graphs of simple functions.

2 Vertical Transformations

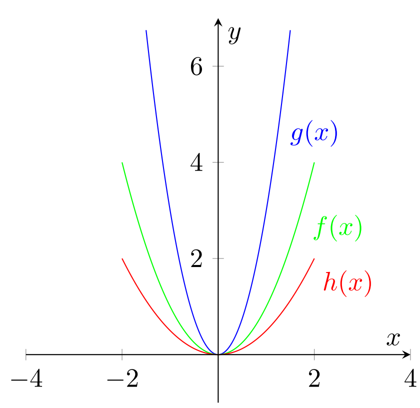
2.1 Vertical Translation

Suppose we have three functions, for example $f(x) = x^2$, $g(x) = x^2 + 1$ and $h(x) = x^2 - 1$. How do we find the graph of g and h? We notice that $g(x) = f(x) + 1$ and $h(x) = f(x) - 1$. Therefore if we shift the points on graph of f, i.e. $(x, f(x))$ up by 1 unit, we get $(x, f(x) + 1)$, which is $(x, g(x))$ i.e. the corresponding point on graph of g. Similarly, to get the graph of h we just shift the graph of f down by 1 unit. We have the following graphs:

Vertical Translation



Vertical Streching



We can also consider the multiplication by a constant, for example, $f(x) = x^2$, the standard parabola, will be stretched or compressed vertically if we consider the graph of $g(x) = 3x^2$ and $h(x) = \frac{1}{2}x^2$. To see this just think about the relation among points (x, x^2), $(x, 3x^2)$ and $(x, \frac{1}{2}x^2)$. See the graph in the previous page.

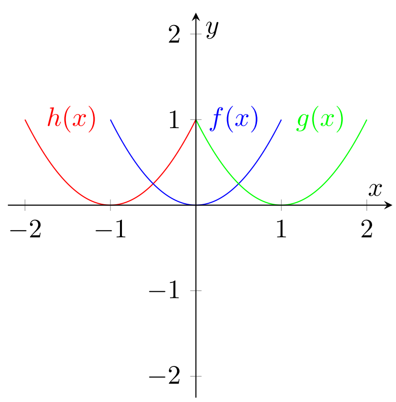
One of the property of vertical transformation is that it does not change the domain of the original function.

2.2 Horizontal Transformation

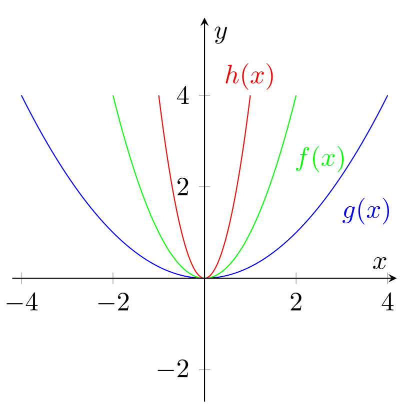
Suppose we still, have three functions, $f(x) = x^2$, $g(x) = (x-1)^2$ and $h(x) = (x+1)^2$. How do we find the graph of g and h? We notice that $g(x) = f(x-1)$ and $h(x) = f(x+1)$. It is not very obvious how should we get the graph of g and h from shifting the graph of f horizontally. Intuition might tell us to shift right to get h and left to get g, but that is not the case.

If we shift the points on graph of f, i.e. $(x, f(x))$ left by 1 unit, we get $(x-1, f(x))$, let $t = x - 1$ be a substitution, the same point should be written as $(t, f(t+1))$ i.e. $(t, h(t))$, the corresponding point on graph of h. Similarly, to get the graph of g we shift the graph of f right by 1 unit. We have the following graphs:

Horizontal Translation



Horizontal Streching



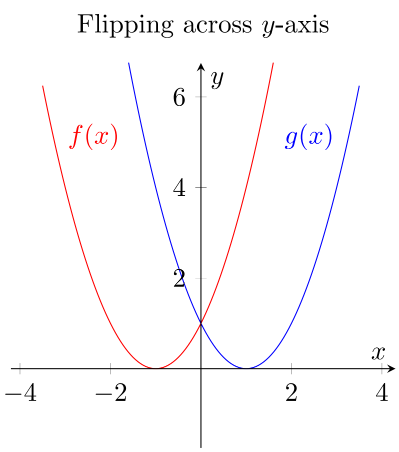
We can also consider the multiplication by a constant, for example, $f(x) = x^2$, the standard parabola, will be stretched or compressed horizontally if we consider the graph of $g(x) = f(\frac{x}{2}) = (\frac{x}{2})^2$ and $h(x) = f(2x) = (2x)^2$. To see this just think about the relation among points (x, x^2), $(x, 3x^2)$ and $(x, \frac{1}{2}x^2)$.

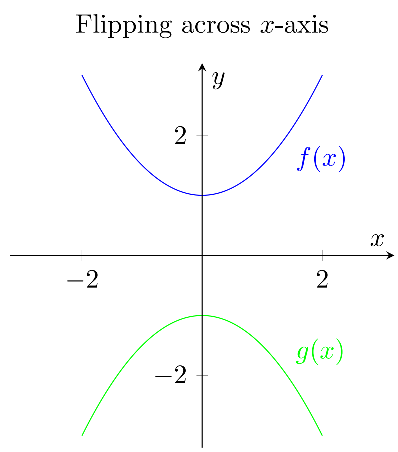
Domain Problem: Horizontal transformation usually changes the domain. For example if we know that the domain of $y = f(x)$ is $[-1, 3]$, then the domain of $y = g(x) = f(2x+1)$ should be calculated as follows: since the domain of g

is the set of x values for $g(x)$ to make sense, by definition we need $2x+1$ to be in $[-1,3]$ for $g(x)=f(2x+1)$ to make sense. Hence we have $1 \le 2x+1 \le 3$, solving it we get $0 \le x \le 1$. Therefore the domain of g is $[0,1]$.

2.3 Flipping Graphs Across Axis

Flipping graphs across x or y axis are also usual transformation of graphs. For example, if $f(x)=x^2+2x+1$ and $g(x)=f(-x)=x^2-2x+1$, then the graph of f and g are symmetric with respect to y axis:





We also have flipping across x axis: for example if $f(x)=\frac{x^2}{2}+1$ and $g(x)=-f(x)=-\frac{x^2}{2}-1$, then the graph of g is obtained by flipping the graph of f across x axis. Generally if we have two functions f and g with $g(x)=-f(x)$ then their graphs have such a relation.

2.4 Odd and Even Functions

There is a basic symmetry of functions associated to the flipping across axis. One is called odd function and the other called even function. Odd functions refers to those which satisfies $f(x)=-f(-x)$, hence if we flip the graph across x-axis and then y-axis, we get the same graph. Even functions refers to those which satisfies $f(-x)=f(x)$, hence it is symmetric with respect to y-axis. For example, $f(x)=x$ is an odd function and $f(x)=x^2$ is even. Sometimes we face the problem of defining a function with some given assumption so that it becomes an odd function or even function. For example,

Example. Let $f(x)=x+1$ for x in $[0,3]$. Try to extend the definition of f to $[-3,3]$ so that it is an even function. Can you make f an odd function?
Solution. To make f an even function we need $f(x)=f(-x)$ for x in $[-3,3]$. Since we know that $f(x)=x+1$ on the right half, then for x in $[-3,0)$, we have

$f(x) = f(-x) = -x + 1$. We have the second equality since if x is in $[-3, 0)$ then $-x$ is in $(0, 3]$ so the original definition of f works there. Therefore

$$f(x) = \begin{cases} -x + 1 & x \text{ in } [-3, 0) \\ x + 1 & x \text{ in } [0, 3] \end{cases}$$

We can not make f an odd function. An odd function requires that $f(-x) = -f(x)$, if we plug in $x = 0$, then $f(-0) = -f(0)$ which is $f(0) = -f(0)$, hence $2f(0) = 0$ and $f(0) = 0$. This is a necessary condition. However by definition of f on $[0, 3]$, $f(0) = 0 + 1 = 1$. Therefore f can not be an odd function anyway.

Further Question. What if we only define f on $(0, 3]$ to be $f(x) = x + 1$? Can we extend that to $[-3, 3]$ such that f is odd?