# Math 3C Section 3.3

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## 1 Basic Manipulations between Two Functions

Just as four algebraic manipulations of real numbers, we can define them for functions analogously:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The potential problem with these manipulations is that the domain of new functions are usually changed, in a non-trivial way. Let us look at the following example.

**Example.** Let $f(x) = \sqrt{x+4}$ and $g(x) = \sqrt{x-2}$, find the unspecified domain of $f/g$.

*Solution.* We have $(f/g)(x) = \sqrt{(x+4)/(x-2)}$ by convention, therefore the unsepcified domain is $(-\infty, -4] \cup (2, \infty)$ by the discussion in preceding lecture notes. However, the original domain of $f$ is $[4, \infty)$ and domain of $g$ is $[2, \infty)$, hence the domain of $f/g$ is neither an intersection or a union of the original domains.

## 2 Composition of Functions

Another way of associating functions is rather unique to the realm of functions, unlike basic manipulations. To composite 2 functions means to form a stream line of different machines, so that the output of the first function automatically becomes an input of the second function. We define $f \circ g$ as follows.

$$(f \circ g)(x) = f(g(x))$$

If there are 3 or more functions we just add each function in some order like

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Just remember that the closest function acts first, hence $(h \circ g \circ f)(x) = h(g(f(x)))$.

Order of functions is an essential factor of the outcome of composition. For example, if $f(x) = x^2 + 1$ and $g(x) = 2x + 4$, then $(f \circ g)(x) = 4x^2 + 16x + 17$ and $(g \circ f)(x) = 2x^2 + 6$.

Another important but implicit change that undergoes during the composition is the domain of functions. For example $f(x) = \sqrt{x-1}$ and $g(x) = 2x$, then $(f \circ g)(x) = \sqrt{2x-1}$. The domain of $f \circ g$ is not inherited directly from the domain of $g$, but is shrunk from the whole real line to $[1/2, \infty)$.

The composition of functions resembles product of real numbers in many perspectives besides the commutativity (that $f \circ g \neq g \circ f$ generally). But we have associatitvity:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

To understand it take the following example, $f(x) = x + 1$, $g(x) = 2x + 5$ and $h(x) = x^2$. Then we have

$$(f \circ g)(x) = 2x + 6, \qquad [(f \circ g) \circ h](x) = 2x^2 + 6$$

and we also have

$$(g \circ h)(x) = 2x^2 + 5, \qquad [f \circ (g \circ h)](x) = 2x^2 + 6$$

We also have functions that act like "0" or "1" as in the multiplication of real numbers. They are constant functions and identity function. A constant function is a function with a constant value, i.e. $f(x) = c$ for all $x$ in the domain of $f$. One can see that when making composition with any other functions, like $g(x) = x^2, x^3 + 2x + 9$ and so on, we have

$$(f \circ g)(x) = c, \quad (g \circ f)(x) = g(c)$$

both $f \circ g$ and $g \circ f$ are constant functions. That is like 0 times everything is just 0. The identity function is the unique $f(x) = x$. When making compositions we have

$$(f \circ g)(x) = (g \circ f)(x) = g(x)$$

so compostion with $f$ does not change the function $g$. That is like 1 times everything does not make any difference.

# 3 Decomposition of functions

This is an inverse problem of composition of functions, that is if we have a complicated function, how can we write it as a composition of several simple functions. For example: $T(x) = \frac{4}{5+x^2}$ To decompose $T$ into $f \circ g \circ h$, we just observe that from $x$ to $T(x)$, we did $x \to x^2 \to x^2 + 5 \to \frac{4}{x^2+5}$, therefore $h(x) = x^2$, $g(x) = x + 5$ and $f(x) = 4/x$.