1 Basic Manipulations between Two Functions

Just as four algebraic manipulations of real numbers, we can define them for functions analogously:

\[(f + g)(x) = f(x) + g(x)\]
\[(f - g)(x) = f(x) - g(x)\]
\[(f \cdot g)(x) = f(x) \cdot g(x)\]
\[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\]

The potential problem with these manipulations is that the domain of new functions are usually changed, in a non-trivial way. Let us look at the following example.

**Example.** Let \(f(x) = \sqrt{x + 4}\) and \(g(x) = \sqrt{x - 2}\), find the unspecified domain of \(f/g\).

**Solution.** We have \((f/g)(x) = \sqrt{(x + 4)/(x - 2)}\) by convention, therefore the unspecified domain is \((-\infty, -4) \cup (2, \infty)\) by the discussion in preceding lecture notes. However, the original domain of \(f\) is \([4, \infty)\) and domain of \(g\) is \([2, \infty)\), hence the domain of \(f/g\) is neither an intersection or a union of the original domains.

2 Composition of Functions

Another way of associating functions is rather unique to the realm of functions, unlike basic manipulations. To composite 2 functions means to form a stream line of different machines, so that the output of the first function automatically becomes an input of the second function. We define \(f \circ g\) as follows.

\[(f \circ g)(x) = f(g(x))\]

If there are 3 or more functions we just add each function in some order like

\[(f \circ g \circ h)(x) = f(g(h(x)))\]
Just remember that the closest function acts first, hence \((h \circ g \circ f)(x) = h(g(f(x)))\).

Order of functions is an essential factor of the outcome of composition. For example, if \(f(x) = x^2 + 1\) and \(g(x) = 2x + 4\), then \((f \circ g)(x) = 4x^2 + 16x + 17\) and \((g \circ f)(x) = 2x^2 + 6\).

Another important but implicit change that undergoes during the composition is the domain of functions. For example \(f(x) = \sqrt{x - 1}\) and \(g(x) = 2x\), then \((f \circ g)(x) = \sqrt{2x - 1}\). The domain of \(f \circ g\) is not inherited directly from the domain of \(g\), but is shrunk from the whole real line to \((1/2, \infty)\).

The composition of functions resembles product of real numbers in many perspectives besides the commutativity (that \(f \circ g \neq g \circ f\) generally). But we have associativity:
\[
(f \circ g) \circ h = f \circ (g \circ h)
\]
To understand it take the following example, \(f(x) = x + 1\), \(g(x) = 2x + 5\) and \(h(x) = x^2\). Then we have
\[
(f \circ g)(x) = 2x + 6, \quad [(f \circ g) \circ h](x) = 2x^2 + 6
\]
and we also have
\[
(g \circ h)(x) = 2x^2 + 5, \quad [f \circ (g \circ h)](x) = 2x^2 + 6
\]
We also have functions that act like "0" or "1" as in the multiplication of real numbers. They are constant functions and identity function. A constant function is a function with a constant value, i.e. \(f(x) = c\) for all \(x\) in the domain of \(f\). One can see that when making composition with any other functions, like \(g(x) = x^2, x^3 + 2x + 9\) and so on, we have
\[
(f \circ g)(x) = c, \quad (g \circ f)(x) = g(c)
\]
both \(f \circ g\) and \(g \circ f\) are constant functions. That is like 0 times everything is just 0. The identity function is the unique \(f(x) = x\). When making compositions we have
\[
(f \circ g)(x) = (g \circ f)(x) = g(x)
\]
so composition with \(f\) does not change the function \(g\). That is like 1 times everything does not make any difference.

3 Decomposition of functions

This is an inverse problem of composition of functions, that is if we have a complicated function, how can we write it as a composition of several simple functions. For example: \(T(x) = \frac{4}{5+x^2}\) To decompose \(T\) into \(f \circ g \circ h\), we just observe that from \(x\) to \(T(x)\), we did \(x \rightarrow x^2 \rightarrow x^2 + 5 \rightarrow \frac{4}{x^2 + 5}\), therefore \(h(x) = x^2, g(x) = x + 5\) and \(f(x) = 4/x\).