Math 3C Section 3.4 & 3.5

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1 Inverse Functions

The study of inverse functions begins with the study of general formula of equations. If we consider the simple equation $2x = 6$, it is obvious that $x = 3$ is the solution as we can divide both sides by 2. If we generalize that method to study a family of equations $2x = a$ (a is any given real number), we have $x = a/2$. It is convenient to think of that relation as a determination of x in terms of a. If $a = 2x$ is regarded as a function of x, then $x = a/2$ is called the inverse function with variable a.

Definition. For a given function $y = f(x)$, if there is a function $y = g(x)$ such that $g(f(x)) = x$ for all x in the domain of f, then g is called the inverse function of f, denoted as $g = f^{-1}$.

Roughly speaking, the inverse function is a function which reverse the role of output and input for the original function. From the definition above we have the following composition rule:

$$f^{-1} \circ f = \mathrm{id}$$

where id is the identity function $\mathrm{id}(x) = x$.

1.1 Invertibility of functions

Remember that a function $y = f(x)$ associates a unique $f(x)$ to every x in its domain. But sometimes different x values correspond to the same $f(x)$, i.e. $f(x_1) = f(x_2)$ holds for some $x_1 \neq x_2$.

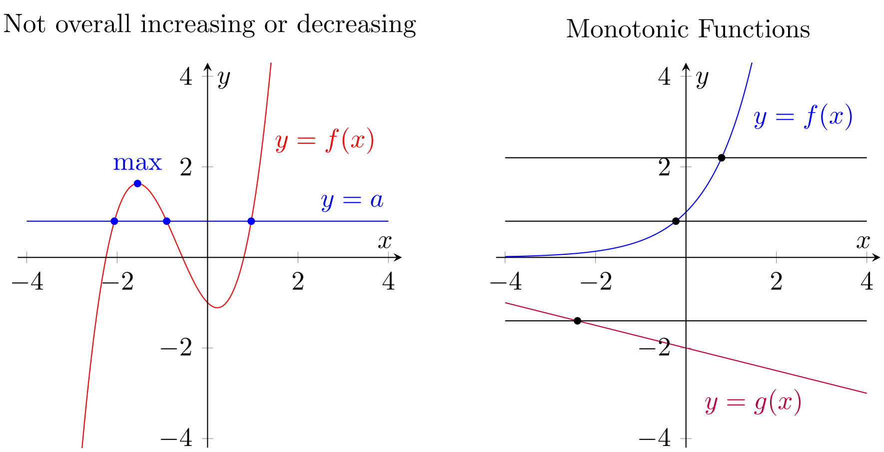
For example, the absolute value function $y = |x|$ takes both $x_1 = -1$ and $x_2 = 1$ to 1. The quadratic function $y = x^2$ takes both $x_1 = -2$ and $x_2 = 2$ to 4. If we try to find the inverse of $f(x) = x^2$, the major difficulty is to define the values of $f^{-1}(1)$, $f^{-1}(4)$ and so on, since they are not unique for the preceding reason. In this case we say such a function is not **invertible**. What concept do we use to characterize the *invertibility* of functions in general?

From the discussion above we know that an invertible function must have the following property: one x corresponds to exactly one y, and one y corresponds to exactly one x. More precisely, we use the hypothetical statement: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. These functions are called "1-1" functions.

By the definition of function there is no 1-many functions. But there could be many-1 functions, such as constant functions and periodic functions. To characterize the "1-1" property geometrically we introduce the "monotonicity" of functions. We have the following observation.

A 1-1 function must be either increasing or decreasing.

The reason is simple: if the function has a turning point then it comes back to a value previously achieved. Look at the following graphs.



An increasing function satisfies the following property: whenever $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$. For a decreasing function: whenever $x_1 \leq x_2$, we have $f(x_1) > f(x_2)$. These properties guarantee the 1-1 property, as any horizontal line only intersects the graph at most one point. Hence we have the following conclusion: every monotonic function is invertible.

1.2 Formula of Inverse Functions

Now we consider a practical problem: to find the inverse of a given function $y = f(x)$. The essential step is to express x in terms of y.

Example 1. Let $f(x) = \sqrt{\frac{x+3}{x-4}}$, find the formula of f^{-1}.
Solution. See the solution of practice midterm 1.

Example 2. Let $f(x) = x^2 + x + 6$ with the domain $(0, +\infty)$. Find the formula of f^{-1}.

Solution. Let $y = x^2 + x + 6$ and we first formally solve x in terms of y. Using the quadratic formula for equation

$$x^2 + x + 6 - y = 0$$

we have

$$x = \frac{-1 \pm \sqrt{(-1)^2 - 4(6-y)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{4y-23}}{2}$$

However we have $\pm$ which makes the expression non-unique. Hence we need to use the domain of f to find out whether to choose $+$ or $-$. The easiest way is to plug in $x = 1$ into $f(x)$. Why? Because $x = 1$ is inside the domain of f and relatively easy to compute. We have $f(1) = 8$. Let's check whether $+$ or $-$ works when $x = 1$ and $y = 8$ are plugged into the formula above.

$$1 = -\frac{1}{2} \pm \frac{\sqrt{4 \cdot 8 - 23}}{2} = -\frac{1}{2} \pm \frac{3}{2}$$

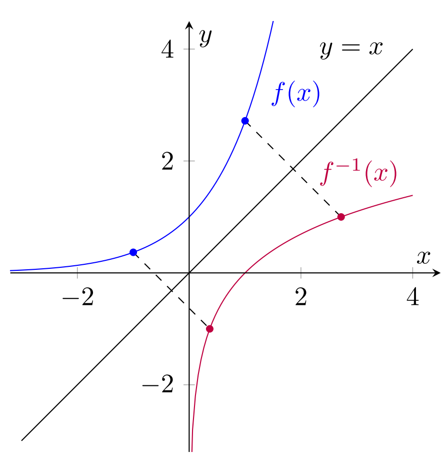
Therefore $+$ is desired. Hence we have

$$f^{-1}(x) = -\frac{1}{2} + \frac{\sqrt{4x-23}}{2}.$$

1.3 Graph of Inverse Functions

Using graphs instead of formulae, we can get a better idea of the behavior of inverse functions. If A point in the form (x, y) is on the graph of f, then (y, x) is the corresponding point on the graph of f^{-1}. To transform (x, y) into (y, x), like $(1, 2) \to (2, 1)$ and $(-3, 5) \to (5, -3)$, we just flip the whole plane across the line $y = x$. Therefore the graph of f^{-1} is the flipping of graph of f across $y = x$.

Graph of inverse Function



From the picture we can see two important conclusions:
(1) If the original function is increasing (or decreasing), then its inverse is also increasing (or decreasing).
(2) The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.