Math 3C Section 4.1 & 4.2

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1 Integer Exponents

We are familiar with numbers expressed in an exponential form: $2^1, 2^2, 2^3, 2^4, 2^5 \cdots$, and their generalization to arbitrary base $x^1, x^2, x^3, x^4, x^5 \cdots$. The basic identity is

$$x^m \cdot x^n = x^{m+n} \quad \text{for natural numbers } m, n$$

Now we want to extend the definition of x^m to where m can be negative integers. First we define $x^0 = 1$ when $x \neq 0$. Then we can see that

$$x^m \cdot x^{-m} = x^{m+(-m)} = x^0 = 1$$

Therefore

$$x^{-m} = \frac{1}{x^m}.$$

We do not try to define 0^0, not because it will violate the our familiar identities, but for some deeper reasons concerning limits and convergence.

Another useful identity is

$$(xy)^m = x^m \cdot y^m \quad \text{for integer } m$$

Using these identities we can simply some complex exponential expressions.

Example. Simplify the following expression.

$$\left(\frac{\left(x^{-3}y^4\right)^2}{x^{-9}y^3} \right)^2$$

Solution.

$$\left(\frac{\left(x^{-3}y^4\right)^2}{x^{-9}y^3} \right)^2 = \left(\frac{x^{-3\cdot 2}y^{4\cdot 2}}{x^{-9}y^3} \right)^2 = \left(\frac{x^{-6}y^8}{x^{-9}y^3} \right)^2 = \left(x^3y^5\right)^2 = x^6y^{10}$$

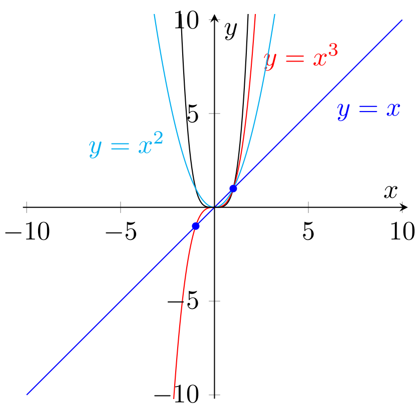
The general rule of simplification is to simplify inner parenthesis first.

1.1 Graph of Power Functions

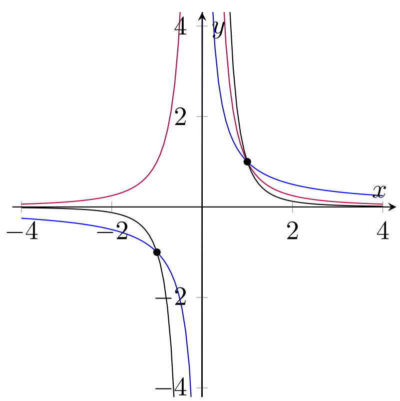
We call $x, x^2, x^3, \ldots, x^n, \ldots$ power functions of x. They play as building blocks of more complicated functions like polynomials, rational functions, and even exponential and trigonometric functions (using infinite sum of powers). It is very important to understand the behavior of power functions. First we study their behavior when x is large.

Generally speaking, when n is an even number, x^n approaches positive infinity when x approaches either $-\infty$ or ∞. When n is an odd number then x^n approaches $-\infty$ as $x \to -\infty$ and $x^n \to \infty$ as $x \to \infty$. These facts can be seen from the following graphs.

Power Functions $n = 1, 2, 3, 4$



Power functions $n = -1, -2, -3$



From the graphs we can see that when n is positive, the growth of function x^n is much faster than that of x^{n-1}. We can see as well the decay of x^{-n} is much faster than x^{-n+1}. Another observation is that if n is an odd number, then x^n and x^{-n} are both odd functions; if n is an even number, then x^n and x^{-n} are both even functions.

2 Polynomials

Polynomials are functions defined as

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are indexed coefficients. They can be any real numbers (including 0). Hence polynomials are natural generalization of power functions $x^n, x^{n-1}, \cdots, x$. Because of the difference of exponents in the components of a polynomial, each single term behaves uniquely, hence making the behavior of their sum well more complex to be tracked. We shall look at two notions associated with every polynomial: degrees and zeros.

2.1 Degree of Polynomials

The degree of a polynomial $p(x)$ is the largest exponent in p. Usually we write an polynomial either in ascending exponents or descending exponents, so it is easy to recognize. For example, if $p(x) = 2x^5 - 3x^3 + 1$, then $\deg(p) = 5$. If $p(x) = 1$, then $\deg(p) = 0$ since constant could be regarded as of exponent 0.

Addition and Subtraction
For the addition and subtraction of polynomials we do have the following conclusion about the change of degrees:

$$\deg(p + q) \leq \max\{\deg(p), \deg(q)\}.$$

$$\deg(p - q) \leq \max\{\deg(p), \deg(q)\}.$$

For example $p(x) = x^2 + x$ and $q(x) = -x^2 + x$ will give sum $(p + q)(x) = 2x$, hence the degree can get lower. If $p(x) = x^2 + x$ and $q(x) = x^2 - x$ will give sum $(p+q)(x) = 2x^2$ where the degree is preserved, therefore we used $\leq$ in the previous formula. For the second conclusion we just note that $-q(x)$ has the same degree as $q(x)$, so the subtraction obeys the same rule as addition.

Multiplication and Composition
For multiplication of polynomials, i.e. $(p \cdot q)(x) = p(x) \cdot q(x)$, the degree obeys a simpler rule:

$$\deg(p \cdot q) = \deg(p) + \deg(q).$$

This property comes from the fact $x^m \cdot x^n = x^{m+n}$, hence the product of leading terms of p and q results in an addition of exponents. For example, if $p(x) = x^2+1$ and $q(x) = x^5 + 3x^3 + 2$, then $(p \cdot q)(x) = x^7 + 4x^5 + 3x^3 + 2x^2 + 2$, where the degrees $7 = 2 + 5$.

For composition of polynomials, the degrees also obey a simple product rule:

$$\deg(p \circ q) = \deg(p) \cdot \deg(q).$$

This comes from the compounded exponents formula $(x^m)^n = x^{mn}$. Using the same example as above, $(p\circ q)(x) = (x^5 + 3x^3 + 2)^2 + 1 = (x^5)^2 + \cdots = x^{10} + \cdots$ whose degree is $10 = 2 \cdot 5$.

2.2 Zeros and Factorization of Polynomials

Like the study of large numbers relies on the fact that they can be factorized into product of smaller prime numbers, like

$$6469693230 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29,$$

it is meaningful to ask whether a large polynomial $p(x)$ can be written as a product of lower degree ones. For example

$$x^7+4x^6+4x^5+10x^4+22x^3+4x^2+9x+18 = (x+2)(x+3)(x^2-x+1)(x^3+2x+3)$$

can make life a lot easier. Are there nice ways to determine the factors of a given polynomial? To avoid trivial stuffs like $x^2 + x + 1 = 1 \cdot (x^2 + x + 1)$, let's give a definition first.

Defintion. A polynomial $q(x)$ is called a **factor** of polynomial $p(x)$ if we can find another polynomial $r(x)$ such that $p(x) = q(x) \cdot r(x)$. To **factorize** $p(x)$ we try to find $q(x)$ and $r(x)$ such that $p(x) = q(x) \cdot r(x)$ and $\deg(q), \deg(r) < \deg(p)$.

To restrict the degrees we want to make sure that $p(x)$ is split into real smaller polynomials. That can be interpreted, from a practical perspective, as pull out at least a degree 1 polynomial like $x - a$ from $p(x)$. How can we fulfill that? Here comes the notion of zeros and a theorem that relates it to a factorization of polynomials.

Defintion. If $p(a) = 0$ for a real number a, then a is called a **zero** of p. In other words, the **zeros** of p is the set of solutions to the equation $p(x) = 0$.

Example. $p(x) = x^2 - 1$ and $q(x) = x^2 + x + 1$, what are the zeros of p and q?

Solution. Setting $p(x) = 0$ we have $x^2 - 1 = 0$, whose solutions are 1 and -1, they are zeros of p. Setting $q(x) = 0$ we have $x^2 + x + 1 = 0$. By the quadratic formula applied with $a = b = c = 1$, we find $\sqrt{b^2 - 4ac} = \sqrt{-3}$ is not a real number. Therefore q has no real zeros.

Now here is a theorem of factorization:

Theorem. If a is a zero of p, then $x - a$ is a factor of $p(x)$. Conversely, if $x - a$ is a factor of $p(x)$ then a is a zero of p.

We can go back to the previous example to see how to apply this theorem. After obtaining $1, -1$ as zeros of $x^2 - 1$, we get its factors $x - 1$ and $x - (-1) = x + 1$:

$$p(x) = (x - 1)(x + 1)$$

But there is no way to factorize $q(x)$ with real polynomials: q has no real zeros.

We conclude this section by a review of factorizing 3 types of polynomials.

Type 1. (Substitutions) Factorize $p(x) = x^6 - 8x^3 + 15$ and $q(x) = x^4 - 2x^2 - 15$.

Solution. Observe that $p(x)$ has only x^6, x^3, x^0 terms, if we replace x^3 by y, then $p(x) = y^2 - 8y + 15$. This is a quadratic polynomial, which can be factorized as $(y - 3)(y - 5)$. Plugging $y = x^3$ back we have

$$p(x) = (x^3 - 3)(x^3 - 5).$$

Similar technique can also be applied to $q(x)$ (let $y = x^2$).

Type 2. (Special Formulae) Here we present some useful formulae:

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots + x^2y^{n-3} - xy^{n-2} + y^{n-1})$$

The formula for $x^n + y^n$ only applies for odd number n. Here is a really special factorization for $x^4 + y^4$:

$$x^4 + y^4 = (x^2 - \sqrt{2}xy + y^2)(x^2 + \sqrt{2}xy + y^2)$$

which does not resemble the formulae above.

Type 3. (Finding Zeros) Some polynomials have simple zeros. For example $p(x) = x^3 - 2x^2 + 4x - 3$ has a zero $a = 1$, therefore $x - 1$ is a factor of $p(x)$. To find the remaining factors, we divide $p(x)$ by $x - 1$, through long division we have

$$x^3 - 2x^2 + 4x - 3 = (x - 1)(x^2 - x + 3)$$

and it turns out that $x^2 - x + 3$ cannot be factorized as $(x - a)(x - b)$ (think about it!), hence that is the best we can do.