Math 3C Section 4.3

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1 Rational Functions

Remember that we have talked about few algebraic manipulations between polynomials: addition(subtraction), multiplication and composition. A common feature of these manipulations is that the result preserves to be a polynomial. However, if we take the quotient of polynomials, i.e. $p(x)/q(x)$, the result is no longer a polynomial in general. It is called a **rational function** which deserves some independent treatment.

Suppose we have the following rational function

$$r(x) = \frac{x^2+1}{x}$$

Then it is quite obvious that we can make some simplification:

$$r(x) = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

to separate the polynomial part x and non-reducible fractional part $1/x$. In general this is usually the first step to get to know a certain rational function. We wish to write every rational function $p(x)/q(x)$ in the following form:

$$G(x) + \frac{R(x)}{q(x)}$$

where G, R are polynomials, especially we require that $\deg(R) \leq \deg(q)$. This resembles the simplification of numerical fractions such as $111/13 = 8 + 7/13$, hence we call $G(x)$ the quotient and $R(x)$ the remainder. As the long division of large whole numbers, we can use long division of polynomials and calculate the following example:

Example. Write $\frac{x^6+3x^3+1}{x^2+2x+5}$ as $G(x) + \frac{R(x)}{q(x)}$.

Solution. After long division(hard to realize in Latex!) we get

$$x^6 + 3x^3 + 1 = (x^4 - 2x^3 - x^2 + 15x - 25)(x^2 + 2x + 5) - 25x + 126$$

Therefore $G(x) = x^4 - 2x^3 - x^2 + 15x - 25$ and $R(x) = -25x + 126$. Hence the answer is

$$\frac{x^6 + 3x^3 + 1}{x^2 + 2x + 5} = x^4 - 2x^3 - x^2 + 15x - 25 + \frac{-25x + 126}{x^2 + 2x + 5}.$$

1.1 Domain of Rational Functions

Rational functions usually have smaller domains than the whole real line. This is due to the fact that the denominator $q(x)$ should not be zero, therefore the zeros of q are **excluded** from the domain. Hence to find the domain of a rational function $r(x) = p(x)/q(x)$, one only need to find the zeros of q.

Example. Find the domains of

$$r_1(x) = \frac{x^6 + 3x^3 + 1}{x^2 + 2x + 5} \quad \text{and} \quad r_2(x) = \frac{x^6 + 3x^3 + 1}{x^2 - 2x - 3}.$$

Solution. Set $x^2 + 2x + 5 = 0$ we have $b^2 - 4ac = 2^2 - 4 \cdot 5 = -16 < 0$. Therefore $x^2 + 2x + 5$ has no real zero, hence the domain of r_1 is simply all real numbers. Set $x^2 - 2x - 3 = 0$ we have roots $x = 3$ or $x = -1$. Hence the domain of r_2 is

$$(-\infty, -1) \cup (-1, 3) \cup (3, \infty) \qquad \text{or} \qquad \{x \neq -1 \text{ and } x \neq 3\}.$$

1.2 Behavior of Rational Functions as $x \to -\infty$ or ∞

The behavior of rational functions is more complex than polynomials since $q(x)$ can be arbitrarily close to 0 despite that it can not be exactly 0. Discussing general behavior of rational functions can be more confusing than enlightening. We use the following example to introduce a conclusion about what could happen when $x \to -\infty$ or ∞.

Example. Discuss the behavior of

$$r_1(x) = \frac{x^2 + 1}{x} \quad \text{and} \quad r_2(x) = \frac{x}{x^2 + 1}$$

as $x \to -\infty$ or ∞.

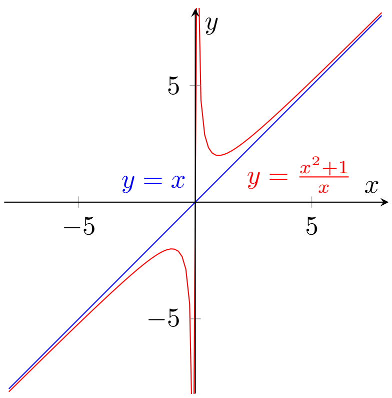
Solution. For r_1 note that we can write it as $x + 1/x$. As $x \to -\infty$, the first term $x \to -\infty$ while the second term $1/x \to 0$. Hence the sum $x + 1/x \to -\infty + 0 = -\infty$. Similar arguments show that when $x \to \infty$, $x + 1/x \to \infty$.

For r_2, note that $r_2(x) = 1/r_1(x)$, hence as discussed the behavior of $r_1(x)$, we see $r_2(x) = 1/r_1(x) \to 1/\infty = 0$.

From the example we see that if $\deg(p) > \deg(q)$, then after the division the quotient $G(x)$ will be a polynomial of degree at least 1, and the behavior of

$p(x)/q(x)$ is determined by the behavior of $G(x)$ when $x \to \infty$ or $-\infty$. If $\deg(p) < \deg(q)$ then it behaves like the reciprocal of the former case, i.e. $p(x)/q(x) \to 1/\infty = 0$.

$\deg(p) > \deg(q)$



$\deg(p) < \deg(q)$