Math 3C Section 5.1

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1 Real Exponents

In Chapter 4 we defined what is called integer exponents and their algebraic properties. Now we want to extend the range of exponents to the largest set of numbers we know: all real numbers. Let us first discuss a relatively easier case: rational exponents, i.e. the meaning of $x^{\frac{m}{n}}$ where m, n are integers.

1.1 Rational Exponents

To get a definition of $x^{\frac{m}{n}}$, it is enough to define $x^{\frac{1}{n}}$ because we can use the product of exponent property to get

$$x^{\frac{m}{n}} = x^{\frac{1}{n}\cdot m} = \left(x^{\frac{1}{n}}\right)^m.$$

To see what should $x^{\frac{1}{n}}$ mean, we take n copies of $x^{\frac{1}{n}}$ and make a product:

$$\underbrace{x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdots x^{\frac{1}{n}}}_{n \text{ copies}} = \left(x^{\frac{1}{n}}\right)^n = x^{\frac{1}{n}\cdot n} = x$$

Therefore we can define $x^{\frac{1}{n}}$ to be $\sqrt[n]{x}$, the n-th radical of x. Hence we are able to see

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = (\sqrt[n]{x})^m$$

or alternatively,

$$x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}.$$

Usually the first way is better for computation, for example:

Example. Compute $25^{\frac{3}{2}}$.
Solution. We calculate in both ways.

$$25^{\frac{3}{2}} = \left(25^3\right)^{\frac{1}{2}} = \sqrt{15625} = 125$$

$$25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3 = \left(\sqrt{25}\right)^3 = 5^3 = 125$$

It is obvious that the second way is easier without using calculators.

Let us take a look at a somewhat unexpected fact with taking square root: sometimes it does not make things worse.

Example. Simplify $(7+4\sqrt{3})^{1/2}$.
Solution. Making use of the complete square formula $(a+b)^2 = a^2+b^2+2ab$, we have

$$a+b = (a^2+b^2+2ab)^{\frac{1}{2}}.$$

Therefore if we can match a^2+b^2 with 7 and $2ab$ with $4\sqrt{3}$ then we are done. It turns out that $a=2$ and $b=\sqrt{3}$ works. Hence we have

$$2+\sqrt{3} = (2^2+(\sqrt{3})^2+2\cdot 2\cdot\sqrt{3})^{\frac{1}{2}} = (7+4\sqrt{3})^{\frac{1}{2}}.$$

1.2 Irrational Exponents

To define x^α where α is an irrational number, i.e. α cannot be represented as m/n (m and n are integers), we can not resort to naturally defined operations like self multiplication or radicals. But through approximation by quantities defined in the preceding sections we can still make sense of irrational exponents. The core idea is to use rational numbers to approximate irrational numbers to an arbitrary precision.

Fact. Every irrational number α can be represented as a decimal fraction with infinitely many digits. Hence if we truncate it at the n-th digit we get a finite decimal fraction which approximates α within an error of 10^{-n}.

Example. $\pi = 3.141592653589793238...$ is a well-known irrational number, and there are infinitely unknown digits of π. However we can approximate π by the following sequence:

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad ... \longrightarrow \pi$$

whose error is approaching 0 as:

$$0.14159\cdots, \quad 0.04159\cdots, \quad 0.00159\cdots, \quad 0.00059\cdots, \quad 0.00009\cdots, \quad ... \longrightarrow 0$$

In the approximating sequence each number is a rational number

$$\frac{3}{1}, \quad \frac{31}{10}, \quad \frac{314}{100}, \quad \frac{3141}{1000}, \quad \frac{31415}{10000}, \quad ... \longrightarrow \pi$$

Therefore we say π can be approximated by a sequence of rational numbers to arbitrary precision.

Definition. We define the irrational exponents x^π as

$$x^\pi \text{ is the limit of } x^3,\ x^{3.1},\ x^{3.14},\ x^{3.141},\ x^{3.1415},\ \cdots$$

And similarly if α is an arbitrary irrational number, write $\alpha = a.bcdefg...$, then

$$x^\alpha \text{ is the limit of } x^a,\ x^{a.b},\ x^{a.bc},\ x^{a.bcd},\ x^{a.bcde},\ \cdots$$

Because of this complicated definition, we usually do not compute irrational exponents by hand. Just keep in mind that once you accept the definition, the algebra of these numbers are the same as integer and rational exponents. Hence

$$x^\alpha \cdot x^\beta = x^{\alpha+\beta}, \quad \left(x^\alpha\right)^\beta = x^{\alpha\beta}, \quad x^{-\alpha} = \frac{1}{x^\alpha} \quad \text{holds for all real } x \text{ and } \alpha.$$

2 Exponential Functions

The preceding discussion leads to a perfectly defined function:

$$f(x) = a^x \qquad a > 0$$

where a is a **given positive real number** and the variable x can be any real number. This function is called **exponential function** with base a.

Question. Why should the base a be a positive number?
Answer. Let us take $a = -1$ which represents negative numbers. Then $(-1)^{\frac{1}{2}}$ is simply $\sqrt{-1}$, not a real number. Hence there is no definition at $x = \frac{1}{2}$. In other words the function $f(x) = (-1)^x$ cannot be defined for all real numbers x. A similar situation happens when $a = 0$, since 0^0 is undefined. These are pathological so we exclude them from the normal exponential functions.

2.1 Graph of Exponential Functions

To discuss the graph of exponential functions, we need to understand their monotonicity (increasing or decreasing) first. If $a > 1$, then from the following fact we observe that $f(x) = a^x$ should be increasing:

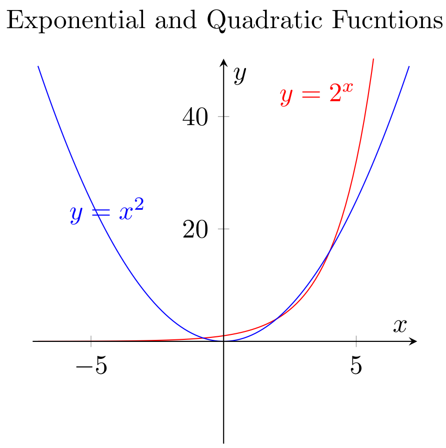
$$1, a, a^2, a^3, a^4, \cdots \text{ is an increasing sequence}$$

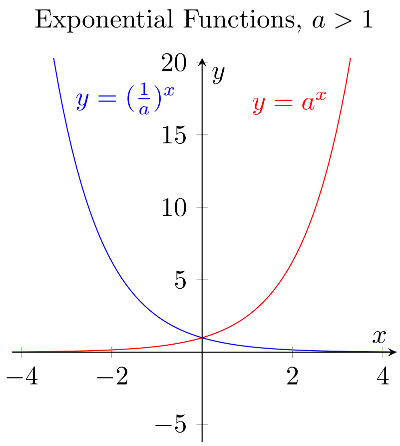
since each term is $a(> 1)$ times the preceding term, hence greater than the preceding one. When $0 < a < 1$, then the sequence above is decreasing, since each term is $a(< 1)$ times the preceding one. Therefore generally we can expect that a^x is increasing function when $a > 1$ and decreasing function when $a < 1$.

Another notable property of the exponential functions are their growth rate. Compare the following table:

x	1	2	3	4	5	6	7	8	9	10	11	12
x^2	1	4	9	16	25	36	49	64	81	100	121	144
2^x	2	4	8	16	32	64	128	256	512	1024	2048	4096

The exponential function 2^x grows much faster than the power function x^2 when x is large. Having these properties in mind we have the following graph on the left:





In the graph on the right we can see the relation between $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$ when $a > 1$. Using $\frac{1}{a} = a^{-1}$ we can see that

$$g(x) = \left(\frac{1}{a}\right)^x = \left(a^{-1}\right)^x = a^{-x} = f(-x)$$

hence the graph of f and g are actually symmetric with respect to y-axis.

Here are some conclusion about the graphs of exponential functions with base $a > 1$:
(1) If $x > 0$ then the graph increases rapidly as $x \to \infty$;
(2) If $x < 0$ then the graph decays rapidly to 0 as $x \to -\infty$;
(3) $(0, 1)$ is always on the graph.

And also about the graphs of exponential functions with base $0 < a < 1$:
(1) If $x > 0$ then the graph decays rapidly to 0 as $x \to \infty$;
(2) If $x < 0$ then the graph increases rapidly as $x \to -\infty$;
(3) $(0, 1)$ is always on the graph.