Math 3C Section 9.1 & 9.2

Yucheng Tu

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1 Unit Circle

The unit circle comes to the stage when we enter the field of trigonometry, i.e. the study of relations among the sides and angles of an arbitrary triangle. The reason is not hard to see: we can easily break a triangle into two right triangles by dropping an altitude(in Figure 1.), and study them individually. And it turns out that the size of these right triangles does not affect their geometric shape, hence we may scale them so that the hypotenuse is 1 and put them into a unit circle(Figure 2.).

If we keep the vertex at the origin but move another vertex along the circle, we get virtually all right triangles. Therefore the unit circle provides a "frame" in which different shapes of right triangles can be compared and synthesized.

1.1 Equation of the Unit Circle

Recalling the definition of the unit circle: the set(locus) of points (x, y) whose distance to the origin $(0, 0)$ is identically 1. Therefore by the distance formula,

$$\text{dist}\big((x, y), (0, 0)\big) = \sqrt{(x-0)^2 + (y-0)^2} = 1$$

Hence the equation of the unit circle is $x^2 + y^2 = 1$. Using this equation we can find the coordinates of points on unit circle when given its either coordinate:

Example. Find x such that $(x, 2/3)$ is on the unit circle.
Solution. Plug $(x, 2/3)$ into the equation of unit circle:

$$x^2 + \left(\frac{2}{3}\right)^2 = 1$$

Hence $x = \pm\sqrt{5}/3$. Therefore there are two points with y-coordinate equal to $2/3$, which are $(-\sqrt{5}/3, 2/3)$ and $(\sqrt{5}/3, 2/3)$.

1.2 Angles of Points on the Unit Circle

To locate a certain point on the unit circle, we can also use angles. Since for every point on the circle we can find a radius connecting it with the center, the angle between the radius and the positive x-axis is unique for that point. Thus there is correspondence between points and angles. If an angle is measured starting at positive x-axis by counterclockwise rotation to the terminal edge, we call it a positive angle θ, if it is measured by clockwise rotation from positive x-axis, we call it a negative angle θ. Hence every point on the unit circle has a positive angle and a negative angle. For example, the point $(0, 1)$ has a positive angle of $90°$ and a negative angle $-270°$.

We can create angles greater than $360°$: just loop around the circle. Each loop corresponds to $360°$, therefore the point with angle θ also has angles $\theta + 360°, \theta + 720°, \ldots$ Also an angle can be very negative like $-1190°$, to reach that angle we simply loop the circle clockwise 3 times and another $-110°$ (Since $-1190 = 3 \times (-360) - 110$). Hence we can talk about an angle whose degree is arbitrary real number.

1.3 Relation of Coordinates and Angles

The relation of coordinates of a point and the angle it correspond to is a fundamental question of trigonometry. Before the study of general case we look at some special points on the circle.

(1) $\theta = \mathbf{90}° \leftrightarrow (x, y) = (\mathbf{0}, \mathbf{1})$
This is obvious as we find the unit circle on the plane. Other related angles are $0° \leftrightarrow (1, 0)$, $180° \leftrightarrow (-1, 0)$ and $270° \leftrightarrow (0, -1)$.

(2) $\theta = \mathbf{45}° \leftrightarrow (x, y) = (\sqrt{\mathbf{2}}/\mathbf{2}, \sqrt{\mathbf{2}}/\mathbf{2})$.
This correspondence comes from the fact that a $45°$ ray from the origin has slope 1, hence its intersection with the circle has equal coordinates $y = x$. Using the equation of unit circle $x^2 + y^2 = 1$ we can show $x = y = \sqrt{2}/2$ gives the solution in the first quadrant. This immediately gives other angles related to $45°$, for example, $135° \leftrightarrow (-\sqrt{2}/2, \sqrt{2}/2)$, $225° \leftrightarrow (-\sqrt{2}/2, -\sqrt{2}/2)$ and $315° \leftrightarrow (\sqrt{2}/2, -\sqrt{2}/2)$.

(3) $\theta = \mathbf{30}° \leftrightarrow (x, y) = (\sqrt{\mathbf{3}}/\mathbf{2}, \mathbf{1}/\mathbf{2})$ and $\theta = \mathbf{60}° \leftrightarrow (x, y) = (\mathbf{1}/\mathbf{2}, \sqrt{\mathbf{3}}/\mathbf{2})$
To see the correspondence we need to use some elementary geometry, but let us memorize it for the present. This is actually restating such a fact: the shorter side of a right triangle with two interior angles $30°$ and $60°$ is $1/2$ of the hypotenuse. If we sketch that right triangle in the unit circle by dropping a perpendicular segment to x-axis from the point on the unit circle, the hypotenuse is the radius 1, hence its shorter side is $1/2$ and the longer one is $\sqrt{3}/2$. Other related angles are $120°$, $150°$, $210°$, $240°$, $300°$ and $330°$.

2 Radians vs. Degrees

Sometimes we find that the degree as a unit pretty inconvenient for calculations. When we deal with problems on arclength and angles, there is a better unit of measurement of angles called radians. First we see how to approach the following problem:

Problem. Consider a circle of radius 2 and centered at the origin $(0, 0)$. Find the length of **both** arcs connecting points $(\sqrt{2}, -\sqrt{2})$ and $(-1, -\sqrt{3})$.

Analysis. for two different points A and B on a circle, there are two arcs connecting them: following the circle from A to B in a clockwise or counterclockwise direction. If the radius is r, the sum of length of these two arcs are the whole circumference $2\pi r$. We call the shorter one a minor arc and major arc for the other. How can we calculate their individual lengths given the position of A and B?

The line of thought is: **the positions determine the angle, the angle determines the arclength.** The angle here refers to the angle between the radii that connect A and B to the center O respectively. How do we find out the angles first?

Solution. Since the radius is 2 instead of 1, these points actually correspond to $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(-1, -\sqrt{3}/2)$ on the standard unit circle (for non-unit circle everything is timed by the actual radius). Therefore the angle of $(\sqrt{2}, -\sqrt{2})$ is $315°$, and the angle of $(-1, -\sqrt{3})$ is $240°$. Hence we get that, the minor arc connecting these points has angle $315° - 240° = 75°$, and the major arc has angle $360° - 75° = 285°$.

The next step is to find out how angle determines arclength. The conceptual reason is that the circle has rotational symmetry, which means that by looking at a circle you can not tell whether it is spinning around its center or just staying still. Hence if we rotate the sector enclosed by two radii r_A and r_B and either arc between A and B, the whole sector will be preserved, as with its metric property: arclength and angles. Therefore a certain angle correspond to a certain arclength and vise versa. The formula gives the arclength in the case θ is measure by degrees:

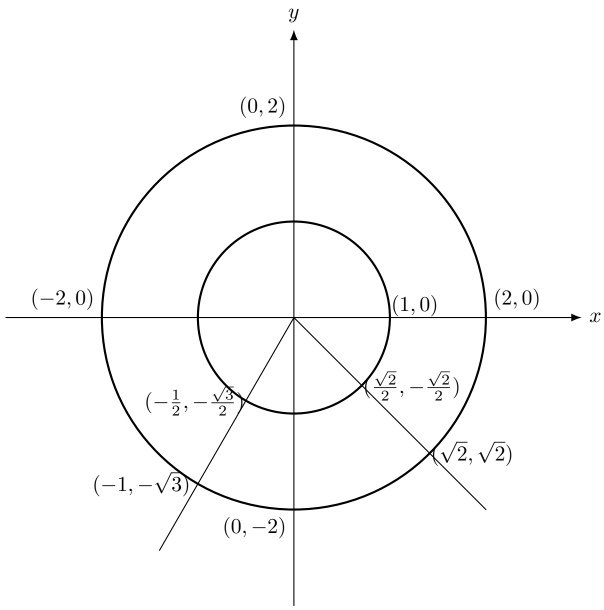
$$l = \mathbf{2\pi r} \cdot \frac{\theta}{\mathbf{360}}.$$

In this formula, $2\pi r$ is the whole circumference and $\theta/360$ is the ratio of the angle with the round angle, which gives the proportion of that arc inside the whole circle. Therefore we have that the length of minor arc is

$$2\pi \cdot 2 \cdot \frac{75}{360°} = \frac{5}{6}\pi$$

and the length of major arc is

$$2\pi \cdot 2 \cdot \frac{285}{360^\circ} = \frac{19}{6}\pi.$$



In the formula of arclength, $\theta/360$ does not seem in harmony with 2π. Since π is a basic constant and we should not abandon it, we introduce the radian system to measure θ, in order to get rid of 360. We will make use of the relation of angle and arclength again, this time on a unit circle.

Definition. The radian of an angle θ is the length of the arc on unit circle opposed to θ. Hence a round angle has radian 2π, a flat angle is π and right angle is $\pi/2$.

The advantage of using radian is immediate: the formula of arclength becomes

$$l = 2\pi r \cdot \frac{\theta}{2\pi}$$

since $\theta/(2\pi)$ is the proportion of θ in the round angle. This formula reduces to

$$l = \theta r$$

since 2π cancel out. This looks simpler than the previous one with degree unit.

Another related question is about the area of a sector with a certain angle θ. We provide the following formula:

$$A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2} r^2.$$

since πr^2 is the area of the circle of radius r. We see the following problem for a fact which may seem false at the first sight:

Problem. There are one 10-inch and one 15-inch pizza. How large a slice do I need to cut off from the 15-inch pizza so that it has the same area as the 10-inch one?
Solution. The problem is asking for the angle of the slice. We can use equation to solve the problem: suppose the desired slice has angle θ, then we have the following equation:

$$\text{Area(10 inch)} = \pi \cdot 10^2 = \frac{\theta}{2} \cdot 15^2 = \text{Area(slice of 15 inch)}$$

Hence

$$\theta = \frac{2 \cdot \pi \cdot 10^2}{15^2} = \frac{8}{9}\pi$$

We only need to cut off less than a half of the 15-inch pizza since $\theta < \pi (= 180^\circ)$. This also means that the difference of 15-in and 10-in is greater than a 10-in itself, although it is not visually obvious.