1 Exponential Function: \( a^x \) \((a > 0)\)

The exponential function is intuitively defined for any natural number \(x\) (multiplication of \(x\) \(a\)'s). It’s increasing if \(a > 1\), decreasing if \(0 < a < 1\), and

\[
\begin{align*}
  a^{x+y} &= a^x + a^y, \\
  a^{xy} &= (a^x)^y. 
\end{align*}
\]

We’d like to extend the domain to all real numbers, while preserving forementioned characteristics. In this regard, we first define nonnegative rational exponents by

\[
\begin{align*}
  a^0 &\triangleq 1, \\
  a^{\frac{1}{x}} &\triangleq \sqrt[a]{x}, \quad x \in \mathbb{N}, \\
  a^{\frac{x}{y}} &\triangleq (a^{\frac{1}{y}})^x = (a^x)^{\frac{1}{y}}, \quad x, y \in \mathbb{N}.
\end{align*}
\]

Then, we extend it to all rational numbers:

\[ a^{-x} \triangleq \frac{1}{a^x}, \quad x \in \mathbb{Q}^+. \]

Finally, the function is continuously extended to reals (note the monotonicity):

\[
a^x \triangleq \begin{cases} 
  \inf\{a^r | r \geq x, r \in \mathbb{Q}\} & \text{if } a \geq 1 \\
  \sup\{a^r | r \geq x, r \in \mathbb{Q}\} & \text{if } 0 < a < 1
\end{cases}, \quad x \in \mathbb{R}.
\]

Here we assume the base \(a\) to be positive. If \(a \leq 0\), we need to be very careful with the domain of the function.

2 Logarithm Function: \( \log_a(x) \) \((a > 0)\)

We often write \(\log_a(x)\) as \(\log(x)\) or \(\ln(x)\), and \(\log_{10}(x)\) as \(\lg(x)\). The logarithm function is the inverse of the exponential function. Since \(a^x\) is from \(\mathbb{R}\) to \(\mathbb{R}^+\), \(\log_a(x)\) is from \(\mathbb{R}^+\) to \(\mathbb{R}\). By the definition of inverse functions, we get

\[
\begin{align*}
\log_a(a^x) &= x, \\
\log_a(a) &= x.
\end{align*}
\]
Using $x = y \Leftrightarrow a^x = a^y$ and properties of exponential functions, we can prove

\[
\log_a(1) = 0, \tag{9}
\]

\[
\log_a(xy) = \log_a(x) + \log_a(y), \tag{10}
\]

\[
\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y), \tag{11}
\]

\[
\log_a(x^y) = y\log_a(x). \tag{12}
\]

### 3 Differentiation

The derivative of $f(x)$ is defined as the limit of the average rate of change (if it exists, or $f$ is differentiable):

\[
f'(x) \triangleq \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]

So the derivative is the instant rate of change of $f(x)$ with respect to $x$.

Formulas for some specific functions:

\[
(x^\alpha)' = \alpha x^{\alpha - 1}, \quad \alpha \in \mathbb{R}, \tag{13}
\]

\[
(\sin(x))' = \cos(x), \tag{14}
\]

\[
(\cos(x))' = -\sin(x), \tag{15}
\]

\[
(a^x)' = a^x \log(a), \tag{16}
\]

\[
(\log_a(x))' = \frac{1}{x \log(a)}. \tag{17}
\]

General derivative formulas:

\[
(k \cdot f(x) + l \cdot g(x))' = k \cdot f(x)' + l \cdot g(x)', \tag{18}
\]

\[
(f(x)g(x))' = f'(x)g(x) + f(x)g'(x), \tag{19}
\]

\[
\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \tag{20}
\]

\[
(f(g(x)))' = f'(g(x))g'(x), \tag{21}
\]

\[
(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}. \tag{22}
\]

### 4 Integration

The integration of $f(x)$ from $a$ to $b$ is defined as the limit of the Riemann sum (if it exists, or $f$ is integrable):

\[
\int_a^b f(x) \, dx \triangleq \lim_{|P| \to 0} S_P(f),
\]

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and an antiderivative of \( f(x) \) is another function \( F(x) \) such that \( F'(x) = f(x) \).

Integration Formulas:
\[
\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx, \tag{23}
\]
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx, \tag{24}
\]
\[
\int_a^b (k \cdot f(x) + l \cdot g(x)) \, dx = k \cdot \int_a^b f(x) \, dx + l \cdot \int_a^b g(x) \, dx, \tag{25}
\]
\[
\int_a^b f(x) \, dx = F(b) - F(a), \tag{26}
\]
\[
\int_a^b f(x) \, dg(x) = \left( f(x)g(x) \right) \bigg|_a^b - \int_a^b g(x) \, df(x), \tag{27}
\]
\[
\int_a^b f(g(x)) \, dg(x) = \int_{g(a)}^{g(b)} f(t) \, dt, \quad (g \text{ monotone}). \tag{28}
\]

5 Taylor Theorem

\( f \in C \) if and only if \( f \) is continuous.

\( f \) is called continuously differentiable, written as \( f \in C^1 \), if \( f \in C \) and \( f' \in C \). Next \( f \) is called twice continuously differentiable, written as \( f \in C^2 \), if \( f \in C^1 \), and \( f'' \in C \). \ldots In General, for \( n \in \mathbb{N}, n \geq 2 \), \( f \) is called \( n^{th} \)-continuously differentiable, written as \( f \in C^n \), if \( f \in C^{n-1} \) and \( f^{(n)} \in C \).

\( f \) is called infinitely continuously differentiable, written as \( f \in C^\infty \), if for any \( n \in \mathbb{N}, f^{(n)} \) exists and is continuous.

The Taylor theorem says that if \( f \in C^\infty \), then
\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots \tag{29}
\]

For example, we can expand \( f(x) = e^x \) around \( x_0 = 0 \). Recall that \( f'(x) = e^x \), it’s easy to observe \( f^{(n)}(x) = e^x \). Plug into the Taylor formula (29), we get
\[
e^x = e^{(0)} + \frac{e^{(0)}}{1!}(x-0) + \frac{e^{(0)}}{2!}(x-0)^2 + \cdots + \frac{e^{(0)}}{n!}(x-0)^n + \cdots
\]
\[
= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots
\]

Further, let \( x = 1 \), we obtain an important expression for the natural number:
\[
e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{30}
\]

Recall that we had learnt another formula for \( e \):
\[
e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x. \tag{31}
\]