Today: ASV 1.5 and 3.1

Next: ASV 3.2

This week:

homework #2 (due Friday Oct, 11 11:59pm)

check 5(a)
E.g. (from last lecture)

An urn has 4 red and 4 white balls. Choose two balls.

\[ W = \{ \text{first ball is white} \} \]
\[ R = \{ \text{second ball is red} \} \]

1) choose balls with replacement

\[ P(W) = \frac{4 \cdot 7}{8 \cdot 7} = \frac{1}{2} \quad P(R) = \frac{7 \cdot 4}{8 \cdot 7} = \frac{1}{2} \]

\[ P(W \cap R) = \frac{4 \cdot 4}{8 \cdot 7} = \frac{2}{7} \neq P(W) \cdot P(R) = \frac{1}{4} \]

\[ \text{conclusion:} \quad W \text{ and } R \text{ are independent} \]

2) choose balls without replacement

\[ P(W) = \frac{4 \cdot 7}{8 \cdot 7} = \frac{1}{2} \quad P(R) = \frac{7 \cdot 4}{8 \cdot 7} = \frac{1}{2} \]

\[ P(R \mid W) = \frac{4}{7} \neq P(R) = \frac{1}{2} \]

(alternatively, \[ P(R \mid W) = \frac{4}{7} \neq P(R) = \frac{1}{2} \]

\[ \text{conclusion:} \quad W \text{ and } R \text{ are not independent} \]
A and B independent $\iff$ A and $B^c$ independent

Proof. $(\Rightarrow)$ Suppose that $A$ and $B$ are independent.

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A) \cdot P(B) = P(A)(1 - P(B))$$

$$= P(A) \cdot P(B^c)$$

Thus, $A$ and $B^c$ are independent.

$$A = (A \cap B^c) \cup (A \cap B)$$

$\hookrightarrow$ disjoint
More than two events?

**Def.** A collection $A_1, \ldots, A_n$ of events is mutually independent if

for any subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

**E.g.** When $n=3$, this means that we must have

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$
$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$
Important example

Toss a coin three times

\[ A = \{ \text{there is exactly 1 Tails in the first two} \} \]

\[ B = \{ \text{there is exactly 1 Tails in the last two} \} \]

\[ C = \{ \text{there is exactly 1 Tails in first and last tosses} \} \]

\[ A = \{ (H, T), (T, H) \} \]

\[ B = \{ (T, H), (T, T) \} \]

\[ C = \{ (H, T), (T, H) \} \]

\[ P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C) \]

\[ P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A \cap C) = P(B \cap C) \]

\[ A \cap B \cap C = \emptyset \]

\[ P(A \cap B \cap C) = 0 \]
Random variables

$(\Omega, \mathcal{F}, P)$-probability space

**Definition.** A (measurable) function $X: \Omega \rightarrow \mathbb{R}$ is called a random variable.

For any $B \subset \mathbb{R}$ we can compute $P(X \in B)$.
**Def.** Let $X$ be a random variable (rv). The probability distribution of $X$ is the collection of probabilities $P(X \in B)$ for all $B \subset \mathbb{R}$.

**Remark.** Strictly speaking, $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ Borel sets.

**Examples**

1) **Coin toss**: $\Omega = \{H, T\}$, $X(H) = 0$, $X(T) = 1$

   $$P(X = 0) = P(H) = \frac{1}{2} = P(X = 1) \quad \text{fair coin}$$

2) **Roll a die**: $\Omega = \{1, 2, \ldots, 6\}$, $X(w) = w$

   For any $1 \leq i \leq 6$ $P(X = i) = \frac{1}{6}$
3) Roll a die twice: $\Omega = \{(i,j): i,j \in \{1, \ldots, 6\}\}$

$X_1((i,j)) = i$ (first number) \hspace{1cm} $X_2((i,j)) = j$ (second number)

for $1 \leq i \leq 6$ \hspace{1cm} $P(X_1 = i) = \frac{1}{6}$ \hspace{1cm} $P(X_2 = i) = \frac{1}{6}$

$S = X_1 + X_2$

$P(S = 2) = \frac{1}{36}$ \hspace{1cm} $P(S = 7) = \frac{6}{36}$

$P(S = 3) = \frac{2}{36}$ \hspace{1cm} $P(S = 8) = \frac{5}{36}$

$P(S = 4) = \frac{3}{36}$ \hspace{1cm} $P(S = 9) = \frac{4}{36}$

$P(S = 5) = \frac{4}{36}$ \hspace{1cm} $P(S = 10) = \frac{3}{36}$

$P(S = 6) = \frac{5}{36}$ \hspace{1cm} $P(S = 11) = \frac{2}{36}$

$P(S = 12) = \frac{1}{36}$
4) Choosing a point from unit disk uniformly at random

\[ \Omega = \{ w \in \mathbb{R}^2 : \text{dist}(w, 0) \leq 1 \} \]

\[ X(w) = \text{dist}(w, 0) \]

For any \( r < 0 \), \( P(X \leq r) = 0 \)

For any \( r > 1 \), \( P(X \leq 1) = 1 \)

For any \( r \in [0, 1] \), \( P(X \leq r) = \frac{\text{size}(D_r)}{\text{size}(D_1)} = \frac{\pi r^2}{\pi} = r^2 \)
Def. Random variable $X$ is a discrete rv if there exists a finite or infinite countable collection of points $\{a_1, \ldots, a_n, \ldots\} \subset \mathbb{R}$ such that $\sum_{i} P(X=a_i) = 1$

Example (lecture 3) Toss a coin until first $T$.

$X =$ total number of tosses.
(Already computed before) for any $i = 1, 2, \ldots$

$$P(X = i) = \frac{1}{2^i}$$

$$\sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \quad (\text{geometric series})$$
Discrete rv \( X \) is completely described by its probability mass function (pmf) \( p_X \) given by

\[
p_X(k) = P(X = k)
\]

for all possible values of \( X \).

\[\begin{array}{ccccccccccc}
  k & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
  P(k) & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & - & - & - & - & - & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \\
\end{array}\]

Ex. \( S = \text{sum of two dice} \)

What if for every \( x \in \mathbb{R} \) \( P(X = x) = 0 \)?
**Probability density function**

**Def.** Let $X$ be a rv. If function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$P(X \leq b) = \int_{-\infty}^{b} f(x) \, dx$$

then $f$ is a probability density function of $X$.

**Remark.** Definition implies that for $B \subseteq \mathbb{R}$

$$P(X \in B) = \int_{B} f_{X}(x) \, dx$$
E.g. Distance to 0 from a random point in a disk

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = P(X \leq r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r > 1 \end{cases} \]

Take derivative wrt \( r \)

\[ f_X(x) = \begin{cases} 0, & r < 0 \\ 2r, & 0 \leq r \leq 1 \\ 0, & r > 1 \end{cases} \]