MATH180A: Introduction to Probability

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Today: ASV 3.3

Next: Review + practice exam

This week: regrade requests HN2 8am-11pm

homework #3 (due Friday Oct, 18 11:59pm)

No homework for next week
Warm-up exercise (from lecture 8)

Let $T \subset \mathbb{R}^2$ be a triangle with vertices at $(0,0), (1,0), (1,1)$.

Choose a point uniformly at random from $T$.

Let $X$ be a r.v. that gives the difference between the first and the second coordinates. $F_X$ - c.d.f. of $X$

$$F_X(s) = \begin{cases} 
0, & s < 0 \\
2s - s^2, & 0 \leq s < 1 \\
1, & s \geq 1 
\end{cases}$$

Compute p.d.f. of $X$

$$f_X(x) = F_X'(x)$$

$$f_X(x) = \begin{cases} 
0, & x < 0 \\
2 - 2s, & 0 \leq x \leq 1 \\
0, & x > 1 
\end{cases}$$
**Expectation and variance**

**Probability law** → c.d.f. | fully describes the random variable.

→ p.m.f./p.d.f.

**Important partial information:**
- Repeat experiment many times
- Identify a set where 'most' of the outcomes are located

**Ex. Toss a coin** $(X \sim \text{Ber}(\frac{1}{2}))$

**Ex. Number of Heads after 100 tosses**

repeat many times $(S_{100} \sim \text{B}(100, \frac{1}{2}))$

$P(S_{100} = 0) = \frac{1}{2^{100}}$  
$P(S_{100} = 50) = \binom{100}{50} \frac{1}{2^{100}} \approx \frac{2^{36}}{2^{100}}$
**Expectation. Discrete case**

**Def.** Let $X$ be a discrete r.v. The expectation of $X$ is defined by

$$E(X) = \sum_k k \cdot P(X = k)$$

where the sum ranges over all possible values $k$ of $X$.

**Expectation** = weighted average of possible outcomes

= center of mass (expectation ≠ most likely outcome)

**Ex.** $X \sim \text{Ber}(\frac{1}{2})$ $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$X \sim \text{Ber}(p)$ $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = 0 \cdot (1-p) + 1 \cdot p = p$
Expectations of discrete r.v.'s. Examples

Example. Binomial distribution

\[ X \sim B(n, p). \]

\[ E(X) = np \]

\[
E(X) = \sum_{k=0}^{n} k \cdot P(X=k) \\
= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
= \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
= \sum_{k=0}^{n} k \cdot \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
= \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
= np \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
= np \sum_{k=0}^{\ell} \frac{(n-1)!}{\ell!(n-1-\ell)!} p^\ell (1-p)^{n-1-\ell} \\
= np \\
= 1
\]

in the sum, denote

\[ \ell := k - 1, \text{ so that } \]

\[ k = \ell + 1, \]

if \[ k = 1, \ell = 0 \]

if \[ k = n, \ell = n - 1 \]
Expectations of discrete r.v.'s. Examples

Example: Geometric distribution

\[ X \sim \text{Geom}(p). \]

\[ E(X) = \frac{1}{p} \]

\[ \sum_{k=0}^{\infty} x_k = \frac{1}{1-x} \]

\[ \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{p} \]

\[ E(X) = \sum_{k=1}^{\infty} k \cdot P(X = k) \]

\[ = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} \]

\[ = p \sum_{k=1}^{\infty} (1-p)^{k-1} = p \cdot D = p \cdot \frac{1}{p^2} = \frac{1}{p} \]

\[ D := \sum_{k=1}^{\infty} (1-p)^{k-1} \]

\[ = \frac{1}{p} + \sum_{\epsilon=0}^{\infty} \epsilon (1-p)^{\epsilon} = \frac{1}{p} + \sum_{\epsilon=1}^{\infty} \epsilon (1-p)^{\epsilon-1} \]

\[ = \frac{1}{p} + (1-p) \sum_{\epsilon=1}^{\infty} \epsilon (1-p)^{\epsilon-1} \]

\[ \implies D = \frac{1}{p} + (1-p)D \implies pD = \frac{1}{p} \]

\[ D = \frac{1}{p^2} \]
Expectation of discrete r.v.'s. Further examples

Let \((\Omega, \mathcal{F}, P)\) be probability space, let \(A \in \mathcal{F}\).

Define \(X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} \) (indicator of event \(A\))

\[
E(\mathbb{1}_A) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = 1 \cdot P(A) = P(A)
\]

Not all r.v.'s have finite expectation:

Example \(X\) is discrete, taking values in \(\{1, 2, \ldots \}\)

\[
P(X = k) = \frac{6}{\pi^2} \frac{1}{k^2}
\]

Is this a well-defined distribution? \(\sum_{k=1}^{\infty} \frac{\zeta}{\pi^2} \frac{1}{k^2} = 1\)

Does it have finite expectation? \(\sum_{k=1}^{\infty} k \cdot \frac{6}{\pi^2} \frac{1}{k^2} = \frac{5}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k} = \infty\)
**Expectation of continuous r.v.'s**

**Def.** Let $X$ be a continuous r.v. with p.d.f. $f_X$.

The expectation of $X$ is defined by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$$

**Remark.** Expectation is also called mean.

**Example**  

$X \sim \text{Unif } [a, b]$.  

$E(X) = ?$  

$$E(X) = \frac{a+b}{2}$$

$$f_X(x) = \begin{cases} 
0, & x < a \\
\frac{1}{b-a}, & a \leq x \leq b \\
0, & x > b
\end{cases}$$

$$\int_{a}^{b} x \cdot \frac{1}{b-a} \, dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$