This week:

- homework #3 (due Friday Oct, 18 11:59pm)

No homework for next week
Warm-up exercise (from lecture 8)

Let $T \subset \mathbb{R}^2$ be a triangle with vertices at $(0,0), (1,0), (1,1)$.

Choose a point uniformly at random from $T$.

Let $X$ be a r.v. that gives the difference between the first and the second coordinates. $F_X$ – c.d.f. of $X$

$$F_X(s) = \begin{cases} 0, & 0 < s < 0 \\ 2s - s^2, & 0 \leq s < 1 \\ 1, & s \geq 1 \end{cases}$$

Compute p.d.f. of $X$
Expectation and variance

Probability law → c.d.f. → p.m.f./p.d.f. | fully describes the random variable.

Important partial information:
- repeat experiment many times
- identify a set where most of the outcomes are located

Ex. Toss a coin \((X \sim \text{Ber}(\frac{1}{2}))\)

Ex. Number of Heads after 100 tosses
repeat many times \((S_{n} \sim \text{B}(100, \frac{1}{2}))\)
Expectation. Discrete case

Def. Let $X$ be a discrete r.v. The expectation of $X$ is defined by

Expectation = weighted average of possible outcomes
= center of mass

Ex. $X \sim \text{Ber}(\frac{1}{2})$  
\[ E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = \frac{1}{2} \]

$X \sim \text{Ber}(p)$  
\[ E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = 0 \cdot (1-p) + 1 \cdot p = p \]
Expectations of discrete r.v.'s. Examples

Example. Binomial distribution

\[ X \sim \text{B}(n, p). \]
Expectations of discrete r.v.'s. Examples

Example: Geometric distribution

\[ X \sim \text{Geom}(p). \]
Expectation of discrete r.v.'s. Further examples

Let $(\Omega, \mathcal{F}, P)$ be probability space, let $A \in \mathcal{F}$. Define $X(w) = I_A(w) = \begin{cases} 1, & w \in A \\ 0, & w \notin A \end{cases}$ (indicator of event $A$)

$$E(I_A) =$$

Not all r.v.'s have finite expectation:

Example $X$ is discrete, taking values in $\{1, 2, \ldots, 4\}$

$$P(X = k) = \frac{6}{\pi^2} \frac{1}{k^2}$$

Is this a well-defined distribution?

Does it have finite expectation?
Expectation of continuous r.v.'s

Def. Let $X$ be a continuous r.v. with p.d.f. $f_X$. The expectation of $X$ is defined by

Remark. Expectation is also called mean.

Example. $X \sim \text{Unif}[a,b]$. $E(X)$ - ?
Expectation of continuous r.v. Examples

Take $X$ r.v. from the warm-up exercise.

p.d.f. $f_X(x) = \begin{cases} 
0, & x < 0 \\
2(1-x), & 0 \leq x \leq 1 \\
0, & x > 1 
\end{cases}$

$E(X) - ?$

Not all r.v.'s have well-defined expectation.
**Theorem.** Let $X$ be a r.v. and let $g$ be a function defined on the range of $X$.

- if $X$ is discrete, then

- if $X$ is continuous, with p.d.f. $f_X$ then
Example

Break a stick of length $l$ at a random point. Let $Y$ be the length of the longer piece.

$E(Y)$ - ?
Another example (insurance policy, lecture 8)

Insurance policy: you pay up to 500 USD, IC pays the rest.
Cost of repair is distributed uniformly on [100, 1500]