MATH180A: Introduction to Probability

www.math.ucsd.edu/~ynemish/180a

Today: ASV 3.3-3.4

Next: ASV 3.5

This week:

homework #4 (due Friday Nov, 1 11:59pm)
Expectation. Discrete case

**Def.** Let $X$ be a discrete r.v. The expectation of $X$ is defined by

\[ E(X) = \sum_{k} k P(X=k) \]

where the sum ranges over all possible values $k$ of $X$.

Expectation of continuous r.v.'s

**Def.** Let $X$ be a continuous r.v. with p.d.f. $f_X$. The expectation of $X$ is defined by

\[ E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx \]

Notation: $E(X) = \mu$
**Expectation of a function of r.v.**

**Thm.** Let $X$ be a r.v. and let $g$ be a function defined on the range of $X$.

- if $X$ is discrete, then $\mathbb{E}(g(X)) = \sum k^n P(X = k)$
- if $X$ is continuous, with p.d.f. $f_X$ then $\mathbb{E}(g(X)) = \int x^n f_X(x) \, dx$

**Important class of functions:** $g(x) = x^n$ "moments"

- discrete: $\mathbb{E}(X^n) = \sum k^n P(X = k)$  
  \( n \)-th moment
- continuous: $\mathbb{E}(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x) \, dx$
Example

Break a stick of length $l$ at a random point. Let $Y$ be the length of the longer piece.

$E(Y) = \; ?$

Let $X \sim \text{Unif}[0, l]$. 

$E(Y) = E(g(X)) = $
Another example (insurance policy, lecture 8)

Insurance policy: you pay up to $500 USD, IC pays the rest.
Cost of repair is distributed uniformly on $[100, 1500]$

Let $X \sim \text{Unif}(100, 1500)$

If $Y = $ your costs, then $Y = g(X)$, where

\[ E(Y) = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx \]

\[ g(x) \]
**Variance**

**Def.** Let $X$ be a r.v. with mean $\mu$.

The **variance** of $X$ is defined by

$$\text{Var} \,(X) =$$

Often denoted as $\sigma^2 := \text{Var} \,(X)$

$$\text{SD}(X) = \sigma = \text{is called the standard deviation}.$$

Variance and standard deviation measure fluctuations of r.v. around its mean.
Computing variance

Let $X$ be a r.v. We can compute $\text{Var}(X)$ as the expectation of $g(x)$, with $g(x) = (x - \mu)^2$

If $X$ is discrete, then

$$\text{Var}(X) = \sum \left( x - \mu \right)^2 p_x(x)$$

If $X$ is continuous with p.d.f. $f_x(x)$

$$\text{Var}(X) = \int (x - \mu)^2 f_x(x) dx$$

Example. $X \sim \text{Ber}(p)$, We know that $E(X) = p$. 

$$\text{Var}(X) = \sum (x - \mu)^2 p_x(x)$$
Alternative formula for the variance

Thm. Let $X$ be a r.v. Then

$$\text{Var}(X) =$$

Proof (for continuous r.v.)

Let $X$ be a continuous r.v. with p.d.f. $f_X$. Then

$$\text{Var}(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x) \, dx =$$
Variance of geometric r.v.

Let $X \sim \text{Geom}(p)$. We know that $E(X) = \frac{1}{p}$.

$$E(X^2) = \sum_{k=1}^{8} k^2 \cdot p(X=k) = \sum_{k=1}^{8} k^2 \cdot p(1-p)^{k-1}$$

$$= 

= 

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Var(X) = \frac{1}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$
Variance of r.v. uniformly distributed on interval

Let $X \sim \text{Unif}[a, b]$. $E(X) = \frac{a+b}{2}$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a} \, dx =$$

$\text{Var}(X^2) =$
Variance of $aX+b$ (affine function)

Let $X$ be a r.v. and $a, b \in \mathbb{R}$. Then

(i) $E(aX+b) =$

(ii) $\text{Var}(aX+b) =$

if $E(X)$ and $\text{Var}(X)$ exist.

Proof
Variance Example.

Let $X$ be a continuous r.v. with p.d.f.

$$f_X(x) = \begin{cases} 3e^{-3x}, & x > 0 \quad (X \sim \text{Exp}(3)) \\ 0, & \text{otherwise} \end{cases}$$

Compute: c.d.f. $F_X$, $P(2 < X < 6)$, $E(X)$, $\text{Var}(X)$, $E(e^{2x})$

(i) $F_X(s) = \begin{cases} 0 \\ \int_0^s \! \! e^{-3x} \, dx \quad \text{for} \quad s \geq 0 \end{cases}$

(ii) $P(2 < X < 6) = \int_2^6 \! \! 3e^{-3x} \, dx$

(iii) $E(X) = \int_0^\infty \! \! x \cdot 3e^{-3x} \, dx$

(iv) $E(X^2) = \int_0^\infty \! \! x^2 \cdot 3e^{-3x} \, dx$

$v) E(e^{2x}) = \int_0^\infty \! \! e^{2x} \cdot 3e^{-3x} \, dx$
Variance Example (Exercise 3.28)

Five closed boxes, three have prizes. Open one after another until you win.

\( X = \) number of prizes opened.

\[ E(X) - ? \quad \text{Var}(X) - ? \]

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<td>( P(X = k) )</td>
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\[ P(X = 1) = \]

\[ P(X = 2) = \]

\[ P(X = 3) = \]

\[ E(X) = \]

\[ \text{Var}(X) = \]