Today: ASV 4.3

Next: ASV 4.4

This week:

homework #4 (due Friday, Nov 1, 11:59pm)
**CLT for the binomial**

**Thm (CLT for binomial).** Let $S_n \sim B(n,p)$, $a < b$. Then

$$
\lim_{n \to \infty} P \left( a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b \right) = \Phi(b) - \Phi(a)
$$

For the average $\frac{S_n}{n}$

$$
P \left( \frac{a \sqrt{p(1-p)}}{\sqrt{n}} \leq \frac{S_n}{n} - p \leq \frac{b \sqrt{p(1-p)}}{\sqrt{n}} \right) \approx \Phi(b) - \Phi(a)
$$

**Rule of thumb:**

$$np(1-p) > 10$$

**Thm (LLN for binomial).** $S_n \sim B(n,p)$. Then for any $\varepsilon > 0$

$$
\lim_{n \to \infty} P \left( \left| \frac{S_n}{n} - p \right| < \varepsilon \right) = 1
$$
Confidence intervals. Motivation

Independent trials, success rate \( p \) (unknown)

\( S_n = \) number of successes after \( n \) trials

\( S_n \sim B(n, p) \). \( \text{E.g.} \) \( (\text{biased}?) \) coin tossed \( n \) times

LLN: \( \frac{S_n}{n} \to p, n \to \infty \) (in probability

If \( n \) is big, \( \frac{S_n}{n} \) is close to \( p \).

\( \frac{S_n}{n} \) \( \sim \) observable, estimate of \( p \)

Usually, we don't know \( p \), but we can get a realization of \( \frac{S_n}{n} \) (flipping coin) for finite \( n \).

What can we say about \( p \)?
Confidence intervals. Set-up

CLT: interval $(-a, a)$

$$P\left(\frac{-a\sqrt{p(1-p)}}{\sqrt{n}} \leq \hat{p} - p \leq \frac{a\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx 2\Phi(a) - 1$$

$$P\left( |\hat{p} - p| \leq \frac{a\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx 2\Phi(a) - 1 = : \gamma$$

Questions:

1) For fixed $n$, find $\varepsilon$ s.t.

$$P(|\hat{p} - p| \leq \varepsilon) \geq \gamma$$

2) For fixed $\varepsilon$, find $n$ s.t.

$$P(|\hat{p} - p| \leq \varepsilon) \geq \gamma$$
Confidence intervals

\[ P \left( r \in [\hat{r} - \varepsilon, \hat{r} + \varepsilon] \right) \geq \gamma \]

\[ [\hat{p} - \varepsilon, \hat{p} + \varepsilon] \] - \( \gamma \)-confidence interval for \( p \)

\( \hat{p} \) is r.v. \( \Rightarrow \) interval is random.

\[ \hat{p} = \frac{S_n}{n} \]

Take some realization of \( \hat{p} \) (a number), say \( \hat{p}_* \)

\[ [\hat{p}_* - \varepsilon, \hat{p}_* + \varepsilon] \] - \( \gamma \)-confidence interval of \( p \)

\( \hat{p}_* \rightarrow \) estimate of \( p \)
Confidence intervals. Computations

- What with unknown $p$ in formula?

$$P\left( |\hat{p} - p| \leq \frac{a \sqrt{p(1-p)}}{\sqrt{n}} \right) \approx 2\Phi(a) - 1 = \gamma$$

$$P\left( |\hat{p} - p| \leq \frac{a \cdot \frac{1}{2}}{\sqrt{n}} \right)$$

The $\gamma$-confidence interval can be taken as

$$[\hat{p} - \varepsilon, \hat{p} + \varepsilon]$$

with

- $\varepsilon \geq \frac{a}{2\sqrt{n}}$ for fixed $n$
- $\sqrt{n} \geq \frac{a}{2\varepsilon}$ for fixed $\varepsilon$

and $2\Phi(a) - 1 \geq \gamma$
Confidence intervals. Example 1 (fixed $n$)

Flip a coin 10,000 times.
Number of Heads is 5,370.
Compute a 99%-confidence interval for $p = P(\text{Heads})$

$n = 10,000$, $\hat{p} = \frac{5,370}{10,000} = 0.537$

$$2\Phi(2\sqrt{10000 \cdot 0.537}) - 1 \geq 0.99$$

$$\Phi(2 \cdot 100 \cdot 0.537) \geq 0.995$$

$$2 \cdot 100 \cdot 0.537 \geq 2.58$$

$$\epsilon \geq \frac{2.58}{2.58} \Rightarrow \epsilon \geq 0.0129$$

99%-confidence interval for $p$ is given by $[0.537 - 0.0129, 0.537 + 0.0129]$
Confidence intervals. Example 2 (fixed accuracy)

Flip a potentially biased coin. How many times should we repeat the experiment to be able to compute a 95\% - confidence interval for \( p = P(\text{Heads}) \) of length 0.01?

\[
\gamma = 0.95, \quad \varepsilon = 0.005
\]

\[
2 \Phi(2 \cdot \gamma \cdot \varepsilon) - 1 = 2 \Phi(2 \cdot 0.005 \cdot \gamma) - 1 \geq 0.95
\]

\[
\Phi(0.01 \cdot \gamma) \geq 0.975
\]

\[
0.01 \cdot \gamma \geq 1.96 \implies n \geq (196)^2
\]

Repeat 196^2 times, compute \( \hat{p} \), \( [\hat{p} - 0.005, \hat{p} + 0.005] \) for \( p \)

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\(-95\%\text{-conf. interval} \)
Confidence intervals. Polling

Without going into details (see Example 4.14) Remark 4.16

Part of population (unknown p) prefers product A / supports candidate B / ...

We interview n individuals and k of them give positive answer (about product / candidate).

What can we say about p?

Fix confidence level \( \gamma \) (often \( \gamma = 0.95 \))

Again, \( \hat{p} = \frac{k}{n} \) is the estimate of p,

\[
\left[ \frac{k}{n} - \varepsilon, \frac{k}{n} + \varepsilon \right]
\]
gives \( 2\Phi(2\varepsilon\sqrt{n}) - 1 \) - confidence interval

\( \Rightarrow \) \( 2\Phi(2\varepsilon\sqrt{n}) - 1 \geq \gamma \rightarrow \) fixed sample size

\( \Rightarrow \) fixed accuracy
Confidence interval. Exercise (rock vs rap)

You ask 400 randomly chosen people living in SD if they prefer rock or rap. 230 reply that they prefer rock music. Give a 99% confidence interval for the part of the population that prefers rock.

- \( n = 400 \), \( \gamma = 0.99 \), \( \hat{p} = \frac{230}{400} = 0.575 \)

\[
2\Phi(2 \sqrt{\frac{1}{400}}) - 1 \geq 0.99 \quad \Rightarrow \quad \Phi(40 \varepsilon) \geq 0.995
\]

\[
\varepsilon \geq \frac{2.58}{40} = 0.0645
\]

\[
[0.575 - 0.0645, 0.575 + 0.0645) \approx 57.5\% \text{ with margin error 6.4%}
\]