Today: ASV 8.4

Next: ASV 9.1-9.2

This week:

- Thanksgiving
- homework 7 due today 11:59 pm
Moment generating function of a sum of independent r.v.'s

Let $X, Y$ be two independent r.v.'s, defined on the same probability space. Then the m.g.f. of $X + Y$

$$E(e^{t(X+Y)}) = E(e^{tX} e^{tY}) = \text{indep} \ E(e^{tX}) E(e^{tY}) \Rightarrow$$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

1) $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, independent. Distribution of $X + Y$?

2) $X \sim \text{N}(\mu_1, \sigma_1^2)$, $Y \sim \text{N}(\mu_2, \sigma_2^2)$, independent. Distribution of $X + Y$?
Covariance

Let \( X \) be a r.v.

- \( E(X) \) - mean value, average of large number of independent realizations

- \( \text{Var}(X) \) - variance, fluctuations of r.v., how far the realizations are spread around the mean

Covariance describes strength and type of dependence between two random variables.

Def. Let \( X \) and \( Y \) be r.v.'s defined on the same probability space. The covariance of \( X \) and \( Y \) is defined by
Covariance

Proposition. \( \text{Cov}(X,Y) = E(XY) - E(X)E(Y) \)

Proof.

Examples. \( X, Y \) discrete r.v. \( P(X=k, Y=l) \) is given in the table. Compute \( \text{Cov}(X,Y) \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0</td>
<td>0.3</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

0.4 0.4 0.2
Some heuristics

By definition, \( \text{Cov}(X, Y) = E((X - E(X))(Y - E(Y))) \)

- \((X - E(X))(Y - E(Y))\) is positive, if \( X - E(X) \) and \( Y - E(Y) \)
- \((X - E(X))(Y - E(Y))\) is negative, if \( X - E(X) \) and \( Y - E(Y) \)

Thus,

\( \text{Cov}(X, Y) > 0 \) means that on average \( X - E(X) \) and \( Y - E(Y) \)

have

\( \text{Cov}(X, Y) < 0 \) means that on average \( X - E(X) \) and \( Y - E(Y) \)

have

\( \text{Cov}(X, Y) = 0 \)
Example (exercise 8.15)
Let \((x, y)\) be uniformly distributed random point on the trapezoid with vertices \((0,0), (2,0), (1,1), (0,1)\).

Is \(\text{Cov}(x, y)\)
(a) positive
(b) negative

\(\text{D}\)
Variance of a sum

Thm. Let $X_1, \ldots, X_n$ be r.v.'s with finite variances.

Then \[ \text{Var} \left( \sum_{i=1}^{n} X_i \right) = \]

If $\forall \ i \neq j \ \text{Cov}(X_i, X_j) = 0$, then \[ \text{Var} \left( \sum X_i \right) = \]

Proof.
Example (8.29)

Urn has 5 balls, 3 red and 2 green. Draw 2 balls and let $X =$ number of red balls. Compute $\text{Var}(X)$ if the sampling is done (a) with replacement (b) without replacement.
Uncorrelated vs independent

1) $X$ and $Y$ are independent $\Rightarrow \text{Cov}(X,Y) = 0$

2) $\text{Cov}(X,Y) = 0 \not\Rightarrow X$ and $Y$ are independent

Proof. 1) If $X$ and $Y$ independent,

2) Enough to find r.v.'s $X$ and $Y$, which are not independent, but $\text{Cov}(X,Y) = 0$
Properties of covariance

Thm. Let $X, X_i, X_2, \ldots, Y, Y_i, Y_2, \ldots$ are r.v.'s defined on the same probability space. Assuming that the covariances below are well-defined, the following hold:

(i) $\text{Cov}(X, Y) =$

(ii) $\text{Cov}(aX + b, Y) =$

(iii) $\text{Cov} \left( \sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} b_j Y_j \right) =$

Proof.
Correlation

Covariance is not good to evaluate the strength of dependence: suppose \( \text{Cov}(X, Y) = 1 \), then \( \text{Cov}(10X, 10Y) = 100 \), but the dependence between \( X \) and \( Y \) is the same as dependence between \( 10X \) and \( 10Y \).

Solution: normalize covariance \( \rightarrow \) correlation

Def. Let \( X, Y \) be r.v., \( \text{Var}(X) < \infty \), \( \text{Var}(Y) < \infty \).

The correlation (coefficient) of \( X \) and \( Y \) is given by
Properties of correlation

Thm. Let \( X, Y \) be r.v. , \( \text{Var}(X) < \infty, \text{Var}(Y) < \infty \). Then

(a) \( \text{Corr}(aX+b, Y) = \)

(b) \( -1 \leq \text{Corr}(X, Y) \leq 1 \)

(c) \( \text{Corr}(X, Y) = 1 \) iff

(d) \( \text{Corr}(X, Y) = -1 \) iff
Example (Exercise 8.54)

Let $X, Y$ be r.v.'s satisfying

$$E(X) = 2, \ E(Y) = 1, \ E(X^2) = 5, \ E(Y^2) = 10, \ E(XY) = 1$$

(a) Compute $\text{Corr}(X, Y)$

(b) Find $c \in \mathbb{R}$ such that $X$ and $X + cY$ are uncorrelated