This week:

- regrade requests: tomorrow 8am - 11pm
- homework #3 (due Friday Oct, 18 11:59pm)
- problem #8 changed
Last time: cumulative distribution function

CDF: for r.v. $X$ 
$F_X(s) = P(X \leq s)$

Discrete r.v.:  
- $F_X(s)$ is a step function (piecewise constant)
- with jumps at possible values of $X$
- magnitude of jump at $x_i \in \mathbb{R} = P(X = x_i)$

Continuous r.v.:  
- $F_X(s) = \int f_X(x) \, dx$, where $f_X(x)$ is pdf of $X$
- $F_X(s)$ is continuous

Examples: 
$X = \text{"roll a fair die"}$

$X = \text{Unif}[1, 6]$
(warm-up) exercise #1.

Family has 3 children of different age. All combinations of boys and girls are equally likely.

\[ X = \text{number of girls in the family} \]

Compute / plot \( F_x \)
(warm-up) exercise #2

Let $T \subset \mathbb{R}^2$ be a triangle with vertices at $(0,0), (1,0), (1,1)$. Choose a point uniformly at random from $T$.

Let $X$ be a r.v. that gives the difference between the first and the second coordinates, i.e., if the chosen point has coordinates $(x,y)$, then $X((x,y)) = x - y$.

Compute/plot $F_X$. 
Relation between c.d.f. and p.m.f./p.d.f.

Thm (P.m.f./p.d.f. from c.d.f.)

Let $X$ be a r.v. and let $F_X$ be its c.d.f.

(a) If $F_X$ is piecewise constant, then $X$ is a discrete r.v.,
the points of discontinuity of $F_X$ correspond to
possible values of $X$, the magnitude of jump at $x$
gives $P(X=x)$;

(b) If $F_X$ is continuous and piecewise differentiable, then
$X$ is a continuous r.v. and its p.d.f. $f_X$ is equal to
$F_X'(x)$ (where the derivative exists).
Not all r.v.'s are purely discrete or purely continuous!

Example (3.20 from ASV)

Insurance policy on a car: in case of an accident

- you pay repairs up to 500 USD,
- if repair costs >500 USD, insurance pays the rest

It is assumed that the costs of repair are distributed uniformly between 100 USD and 1500 USD

Let $X =$ costs YOU pay in case of an accident. $F_X =$ ?
Example 3.20 (cont.)

\[ F_X(s) = \begin{cases} 
0, & s < 100 \\
\frac{s-100}{1500-100}, & 100 \leq s < 500 \\
1, & s \geq 500 
\end{cases} \]
General properties of c.d.f.'s

**Thm** If $F_X$ is a c.d.f. of a r.v. $X$, then

(i) $F_X$ is non-decreasing:

(ii) $F_X$ is right-continuous:

(iii) $\lim_{s \to -\infty} F_X(s) = \lim_{s \to +\infty} F_X(s) = 

\{\text{probability laws}\} \leftrightarrow \text{one-to-one on } \mathbb{R} \leftrightarrow \{\text{functions satisfying (i)-(iii)}\}
Independent trials. Set-up

Recall: discrete r.v.'s $X_1, \ldots, X_k$ are independent iff

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) =$$

$$P\left(\bigcap_{i=1}^{k} \{X_i = x_i\}\right)$$

Assume that an experiment is repeatable (important requirement for scientific experiments)

$\Rightarrow$ sequence of experiments (trials) that are

- modelled by identical probability spaces
- independent
Independent trials of success/failure

Sequence of experiments

**Common probability space**: Product space

define \((\Omega, \mathcal{F}, P)\) by

for \(\omega=(\omega_1, \omega_2, \ldots) \in \Omega\)

\[
P((\omega_1, \omega_2, \ldots)) = \]

The simplest situation:
only two outcomes

Random variables: indicators of success
Bernoulli random variable (parameter $p$)

Each of $X_i$'s above is called Bernoulli r.v.

**Def.** We say that r.v. $X$ has Bernoulli distribution with success rate $p$, $p \in [0,1]$, if

In this case we write $X \sim Ber(p)$

Binomial random variable (parameters $n$ and $p$)

Gives the number of successful trials in the sequence of $n$ independent "Bernoulli experiments".

Let $X_1, \ldots, X_n$ independent r.v., for each $i$ $X_i \sim Ber(p)$.

Denote $S_n := X_1 + \ldots + X_n$. What are possible values of $S_n$?

$S_n$.

Compute $P(S_n = k)$ =
Binomial r.v. Example

Sn above is a binomial r.v. with parameters n and p.

Def. For n ∈ N and 0 ≤ p ≤ 1, we say that r.v. X has binomial distribution with parameters n and p, denoted $X \sim B(n, p)$, if

Example Choose a point uniformly at random from a unit disk. You win if you are at distance ≤ ½ to the center. You play this game 6 times. What is the probability that you win 4 or 5 times out of 6?
Geometric distribution (with parameter $p$)

Gives the number of the first successful trial.

May require infinitely many trials → infinite product space. Probability measure cannot be defined by indep. pointwise

Let $X_1, X_2, \ldots$ infinite sequence of independent Bernoulli with success rate $p$, $X_i \sim \text{Ber}(p)$. Denote by $N$ the number of the first success.

$P(N = k) = \text{Def. Let } p \in (0,1). \text{ We say that r.v. } X \text{ has geometric distribution with parameter } p \text{ if for any } k \in \mathbb{N}$

$P(X = k) = (1-p)^{k-1}p$ Denote $N \sim \text{Geom}(p)$