1. (25 points) 400 randomly chosen individuals were interviewed to estimate the level of support of Yuriy N. as a candidate at the future presidential elections in one of the European countries. Out of 400 interviewed, 168 said that they support Yuriy N. Find the 95% confidence interval for the true value \( p \) of the support of Yuriy N. among the country population. [Hint. You can find the table of values of \( \Phi(x) \) on the last page of this exam].

Solution.

• The 95% confidence interval for the unknown parameter \( p \) is of the form

\[
[\hat{p}_* - \varepsilon, \hat{p}_* + \varepsilon],
\]

where

\[
\hat{p}_* = \frac{168}{400} = \frac{42}{100} = 0.42
\]

and \( \varepsilon \) is the smallest value satisfying

\[
2\Phi(2\varepsilon\sqrt{400}) - 1 \geq 0.95.
\]

• We rewrite the above inequality as

\[
\Phi(2\varepsilon\sqrt{400}) \geq 0.975,
\]

from which, using the table of values of \( \Phi(x) \), we get that

\[
2\varepsilon \cdot 20 = 1.96.
\]

• We conclude from (5) that

\[
\varepsilon = \frac{1.96}{40} = \frac{0.49}{10} = 0.049,
\]

so the 95% confidence interval is

\[
[0.42 - 0.049, 0.42 + 0.049] = [0.371, 0.469].
\]
2. Let $X$ be a random variable having exponential distribution with expectation 2 (i.e., $E(X) = 2$).

(a) (5 points) Compute

$$
\int_{0}^{\infty} e^{-\frac{x^2}{2}} dx. \tag{8}
$$

**Hint.** In this problem you can use without proof the Gaussian integral

$$
\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}. \tag{9}
$$

**Solution.**

- Function

$$
e^{-\frac{x^2}{2}} \tag{10}
$$

is even, therefore

$$
\int_{0}^{+\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{0} e^{-\frac{x^2}{2}} dx. \tag{11}
$$

- We conclude that

$$
\int_{0}^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{\sqrt{\pi}}{2}. \tag{12}
$$

(b) (20 points) Compute

$$
E\left(\frac{1}{\sqrt{X}}\right). \tag{13}
$$

**Solution.**

- If $X \sim \text{Exp}(\lambda)$, then

$$
E(X) = \frac{1}{\lambda}, \tag{14}
$$

thus $X$ has exponential distribution with rate $\frac{1}{2}$, $X \sim \text{Exp}(\frac{1}{2})$.

- For $X \sim \text{Exp}(\frac{1}{2})$

$$
E\left(\frac{1}{\sqrt{X}}\right) = \int_{0}^{+\infty} \frac{1}{\sqrt{x}} \cdot \frac{1}{2} e^{-\frac{x}{2}} dx. \tag{15}
$$

- In order to compute the above integral, apply the change of variable

$$
s = \sqrt{x}, \tag{16}
$$

leading to

$$
\int_{0}^{+\infty} \frac{1}{\sqrt{x}} \cdot \frac{1}{2} e^{-\frac{x}{2}} dx = \int_{0}^{+\infty} e^{-\frac{s^2}{2}} ds. \tag{17}
$$

- Using the result of part (a) of the problem, we get that

$$
E\left(\frac{1}{\sqrt{X}}\right) = \sqrt{\frac{\pi}{2}}. \tag{18}
$$
3. (25 points) Let $X$ be a continuous random variable with p.d.f. given by

$$f_X(x) = \begin{cases} \frac{1}{2x} & x \in (1, e^2), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Let $Y = \ln X$.

(a) (5 points) What is the range (set of possible values) of $Y$?

**Solution.**

Random variable $X$ takes values in the interval $(1, e^2)$. Function $\ln$ is a strictly increasing function, therefore, $Y = \ln X$ takes values in the interval $(\ln 1, \ln e^2) = (0, 2)$.

(b) (15 points) Compute the c.d.f. of $Y$.

**Solution.**

- For a random variable with p.d.f. given by (19), the c.d.f. of $Y = \ln X$ for $s \in (0, 2)$ is given by

$$F_Y(s) = P(Y \leq s) = P(\ln X \leq s). \tag{20}$$

- Function $\ln$ is strictly increasing, so

$$P(\ln X \leq s) = P(X \leq e^s) = \int_1^{e^s} \frac{1}{2x} dx = \frac{1}{2} \ln(x) \bigg|_1^{e^s} = \frac{1}{2} \ln(e^s) - 0 = \frac{s}{2}. \tag{21}$$

- The c.d.f. of $Y$ is given by

$$F_Y(s) = \begin{cases} 0, & s < 0, \\ \frac{s}{2}, & 0 \leq s < 2, \\ 1, & s \geq 2. \end{cases} \tag{22}$$

(c) (5 points) Compute the p.d.f. of $Y$.

**Solution.**

Differentiate the c.d.f. of $Y$ to get that

$$f_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{1}{2}, & 0 \leq y < 2, \\ 0, & y \geq 2, \end{cases} \tag{23}$$

i.e., $Y \sim \text{Unif}[0, 2]$. 

4. (25 points) You are waiting for a (free) bus and you believe that the waiting time (in minutes) has exponential distribution with expectation 10. Alternatively, you can take Uber for $u$ dollars. Both bus and Uber will take you to the destination in 25 minutes, but Uber requires 5 more minutes to find the driver. If you arrive at the destination within next 30 minutes, you will be able to buy a jersey of your favorite basketball team with a 15 dollar discount.

(a) (15 points) What is the expected gain if you decide to wait for the bus?

Solution.

- If you wait for the bus and the bus arrives in less that 5 minutes, you will manage to get to the destination before the sale is over and save 15 dollars. If you wait for the bus for more that 5 minutes, you will be late for the sale and you will save 0 dollars. Therefore, if $X$ is the waiting time for the bus and $Y$ is your gain, $Y$ can be represented as a function of $X$ in the form

$$Y = \begin{cases} 
15, & X < 5, \\
0, & X \geq 5. 
\end{cases}$$

(b) (10 points) For which price range of $u$ will it be more reasonable to take Uber?

Solution.

- If you take Uber for $u$ dollars, you will definitely arrive before the end of the sale and your gain will be $(15 - u)$ dollars in total.
- It might be reasonable to take Uber if your gain will be at least as big at the expected gain from taking the bus, i.e.,

$$15 - u \geq 15 - \frac{15}{\sqrt{e}},$$

or

$$u \leq \frac{15}{\sqrt{e}}.$$