

MATH 142A: Introduction to Analysis

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Today: Subsequential limits

> Q&A: February 1

Next: Ross § 14

Week 5:

- Homework 4 (due Sunday, February 7)
- Quiz 3 (Wednesday, February 3) - Lectures 8-9
- Midterm 1 regards (Monday, February 1 - Tuesday, February 2)
- Homework 2 regards (Monday, February 1 - Tuesday, February 2)

Subsequential limits

Def 11.1 Let (s_n) be a sequence of real numbers and let

$1 \leq n_1 < n_2 < \dots < n_k < \dots$ be an increasing sequence of natural numbers.

Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \dots)$ is called a **subsequence** of $(s_n)_{n=1}^{\infty}$.

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \dots \right)$$
$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \dots \right)$$

Def 11.6 Let (s_n) be a sequence in \mathbb{R} . A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n)

Example • $a_n = (-1)^n, (-1, 1, -1, 1, \dots)$ $\lim_{k \rightarrow \infty} a_{2k-1} = -1, \lim_{k \rightarrow \infty} a_{2k} = 1$

Example • $b_n = 2^{n-1}, \left(\frac{1}{2}, 2, \frac{1}{2^3}, 2^4, \dots \right)$ $\lim_{k \rightarrow \infty} b_{2k-1} = \lim_{k \rightarrow \infty} \frac{1}{2^{2k-1}} = 0$

$$\lim_{k \rightarrow \infty} b_{2k} = \lim_{k \rightarrow \infty} 2^{2k} = +\infty$$

Subsequential limits and \liminf / \limsup

Thm 11.7 Let (s_n) be a sequence. Then there exist

- (i) a monotonic subsequence of (s_n) that converges to $\limsup s_n$
- (ii) a monotonic subsequence of (s_n) that converges to $\liminf s_n$

Proof. If (s_n) is not bounded above, then $\limsup s_n = +\infty$. And by

Thm 11.2 (ii) there exist a subsequence of (s_n) that diverges to $+\infty$.

Suppose (s_n) is bounded above, $\limsup_{n \rightarrow \infty} s_n = t \in \mathbb{R}$

By Thm 11.2 (i) there exists a monotonic subsequence of (s_n) that converges to t iff $\forall \varepsilon > 0$ the set $\{n: |s_n - t| < \varepsilon\}$ is infinite. Fix $\varepsilon > 0$

$$\limsup_{n \rightarrow \infty} s_n = t \Rightarrow \exists N \forall k > N \sup\{s_n: n > k\} < t + \varepsilon \Rightarrow \forall n > N+1 (s_n < t + \varepsilon)$$

Suppose that $\{n: t - \varepsilon < s_n < t + \varepsilon\}$ is finite. Then $\exists N_1 > N$ s.t.

$$\forall n > N_1 (s_n \leq t - \varepsilon) \Rightarrow \sup\{s_n: n > N_1\} \leq t - \varepsilon \Rightarrow \limsup_{n \rightarrow \infty} s_n \leq t - \varepsilon$$

● contradiction

Subsequential limits and convergence

Thm 11.8. Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n) . Then

- (i) S is nonempty (follows Thm 11.7)
- (ii) $\sup S = \limsup s_n$, $\inf S = \liminf s_n$
- (iii) $\lim s_n$ exists iff S has only one element, $S = \{ \lim s_n \}$

Proof (iii) follows from (ii) and Thm 10.7 ($\lim s_n = t \Leftrightarrow \limsup s_n = \liminf s_n = t$)

(ii) Suppose $t \in S \Leftrightarrow \exists$ subsequence (s_{n_k}) s.t. $\lim_{k \rightarrow \infty} s_{n_k} = t$

Then by Thm 10.7 $\liminf_{k \rightarrow \infty} s_{n_k} = \limsup_{k \rightarrow \infty} s_{n_k} = t$

Note that $\forall k (n_k \geq k)$, therefore $\forall N \{s_n : n > N\} \supseteq \{s_{n_k} : k > N\}$

and $\forall N \inf \{s_n : n > N\} \leq \inf \{s_{n_k} : k > N\} \leq \sup \{s_{n_k} : k > N\} \leq \sup \{s_n : n > N\}$

$$\begin{array}{ccccccc} N \rightarrow \infty & & & & & & \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ & \liminf_{S} s_n & \stackrel{9.12}{\leq} & \liminf s_{n_k} = t = \limsup s_{n_k} & \stackrel{9.12}{\leq} & \limsup_{S} s_n & \\ & & & & & \uparrow \text{Thm 11.7} & \end{array}$$

Examples

For each sequence below let S denote the set of subsequential limits.

• $a_n = (-1)^n$

① $S = \{-1, 1\}$

$$\lim_{k \rightarrow \infty} a_{2k-1} = -1, \quad \lim_{k \rightarrow \infty} a_{2k} = 1 \Rightarrow \{-1, 1\} \subset S$$

$$\text{If } t \notin \{-1, 1\}, \text{ then } \forall n \in \mathbb{N} \quad |a_n - t| \geq \min\{|t-1|, |t-(-1)|\} > 0 = t \notin S \\ \Rightarrow S \subset \{-1, 1\}$$

② $\limsup a_n = 1$ $\liminf a_n = -1$

• $b_n = 2^{n(-1)^n}$

① $S = \{0, +\infty\}$

$$\lim_{k \rightarrow \infty} b_{2k-1} = 0, \quad \lim_{k \rightarrow \infty} b_{2k} = +\infty \Rightarrow \{0, +\infty\} \subset S$$

$$\text{If } t \in \mathbb{R}, t \neq 0, \text{ then } \left. \begin{array}{l} \exists N_1 \forall k > N_1 \quad b_{2k-1} < \frac{t}{2} \\ \exists N_2 \forall k > N_2 \quad b_{2k} > t+1 \end{array} \right\} \Rightarrow |b_n - t| > \min\{\frac{t}{2}, 1\} > 0 \quad \forall n > 2 \cdot \max\{N_1, N_2\}$$

② $\limsup b_n = +\infty$, $\liminf b_n = 0$

$$S \subset \{0, +\infty\}$$

The set of subsequential limits is closed

Thm 11.9 Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n) . Then

Let (t_n) be a sequence in $S \cap \mathbb{R}$, i.e. $\forall n (t_n \in S \cap \mathbb{R})$.

If (t_n) has a limit, then $\lim_{n \rightarrow \infty} t_n \in S$.

$$x_n = \frac{1}{2^n}, \forall n \quad x_n \in (0,1), \lim_{n \rightarrow \infty} x_n = 0$$



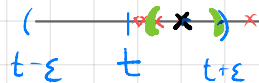
Proof. Suppose $\lim_{n \rightarrow \infty} t_n = t \in \mathbb{R}$. Then

Thm 11.2

$t \in S \Leftrightarrow \exists$ subsequence of (s_n) that converges to $t \Leftrightarrow \forall \varepsilon > 0 \{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ infinite

Fix $\varepsilon > 0$. Then $\exists n_0$ st. $t - \varepsilon < t_{n_0} < t + \varepsilon$

Since $t + \varepsilon - t_{n_0} > 0$, $t_{n_0} - (t - \varepsilon) > 0$



$\delta := \min\{t + \varepsilon - t_{n_0}, t_{n_0} - (t - \varepsilon)\} > 0$ and $(t_{n_0} - \delta, t_{n_0} + \delta) \subset (t - \varepsilon, t + \varepsilon)$

$t_{n_0} \in S$ (subsequential limit) $\stackrel{\text{Thm 11.2}}{\Rightarrow} \{n : |s_n - t_{n_0}| < \delta\}$ is infinite $\Rightarrow \{n : |s_n - t| < \varepsilon\}$ is infinite ■

limsup's and liminf's

Thm 12.1 Let (s_n) and (t_n) be two sequences. Then

$$\left((s_n) \text{ converges } \wedge \lim_{n \rightarrow \infty} s_n = s > 0 \right) \Rightarrow \limsup (s_n \cdot t_n) = s \cdot \limsup t_n$$

Convention: For any $s \in \mathbb{R}$, $s > 0$, $s \cdot (+\infty) = +\infty$, $s \cdot (-\infty) = -\infty$.

Proof

$$\textcircled{1}: \limsup (s_n t_n) \geq s \cdot t \quad (\text{only for } \limsup t_n = t \in \mathbb{R})$$

$$\text{Thm. 11.7} \Rightarrow \exists (t_{n_k}) \text{ such that } \lim_{k \rightarrow \infty} t_{n_k} = t \quad \left| \begin{array}{l} \text{Thm 9.4} \\ \Rightarrow \end{array} \right.$$

$$\text{Thm 11.3} \Rightarrow \lim_{k \rightarrow \infty} s_{n_k} = s.$$

$$\lim_{k \rightarrow \infty} (s_{n_k} t_{n_k}) = s \cdot t$$

$\Rightarrow s \cdot t$ is a subsequential limit for $(s_n t_n) \Rightarrow \limsup (s_n t_n) \geq s \cdot t$ Thm 11.8

$$\textcircled{2}: \limsup (s_n t_n) \leq s \cdot t \quad (\text{only for } s_n > 0 \forall n). \quad \text{Thm 9.5} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s} > 0$$

$$\text{Then } t = \limsup t_n = \limsup \frac{1}{s_n} (s_n \cdot t_n) \stackrel{\textcircled{1}}{\geq} \frac{1}{s} \limsup (s_n \cdot t_n) \Rightarrow$$

$\textcircled{1} \textcircled{2}$

$$\Rightarrow \limsup (s_n t_n) = \limsup (t_n) \cdot s$$

$$s t \geq \limsup (s_n t_n)$$

Remark

If (s_n) and (t_n) are two sequences, and $\lim_{n \rightarrow \infty} s_n = 0$, then there is nothing we can say in general about $\limsup(s_n t_n)$.

- $s_n = \frac{1}{n}$, $t_n = n \Rightarrow \limsup \frac{1}{n} \cdot n = 1$
- $s_n = \frac{1}{n^2}$, $t_n = n \Rightarrow \limsup \frac{1}{n^2} \cdot n = 0$
- $s_n = \frac{1}{n}$, $t_n = n^2 \Rightarrow \limsup \frac{1}{n} \cdot n^2 = +\infty$

Also it is important that one sequence converges.

- $s_n = (0, 1, 0, 1, 0, 1, \dots)$ $\limsup s_n = 1$
 $t_n = (1, 0, 1, 0, 1, 0, \dots)$ $\limsup t_n = 1$ but $\forall n \ s_n \cdot t_n = 0$
- $s_n = (-1)^n$, $t_n = (-1)^{n+1}$ $\limsup s_n = \limsup t_n = 1$, but $\forall n \ s_n \cdot t_n = (-1)^{2n+1} = -1$