MATH 142A: Introduction to Analysis

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Today: Subsequential limits

> Q&A: February 1

Next: Ross § 14

Week 5:

- Homework 4 (due Sunday, February 7)
- Quiz 3 (Wednesday, February 3) - Lectures 8-9
- Midterm 1 regardes (Monday, February 1 - Tuesday, February 2)
- Homework 2 regardes (Monday, February 1 - Tuesday, February 2)

Subsequential limits

Def 11.1 Let (s_n) be a sequence of real numbers and let

$1 \le n_1 < n_2 < \cdots < n_k < \cdots$ be an increasing sequence of natural numbers.

Then $(s_{n_k})_{k=1}^{\infty} = (s_{n_1}, s_{n_2}, s_{n_3}, \ldots)$ is called a subsequence of $(s_n)_{n=1}^{\infty}$.

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \ldots\right)$$

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \ldots\right)$$

Def 11.6 Let (s_n) be a sequence in $\mathbb{R}$. A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n)

Example • $a_n = (-1)^n$, $(-1, 1, -1, 1, \ldots)$

Example • $b_n = 2^{n(-1)^n}$, $\left(\frac{1}{2}, 2^2, \frac{1}{2^3}, 2^4, \ldots\right)$

Subsequential limits and liminf / limsup

Thm 11.7 Let (s_n) be a sequence. Then there exist

(i) a monotonic subsequence of (s_n) that converges to $\limsup s_n$

(ii) a monotonic subsequence of (s_n) that converges to $\liminf s_n$

Proof. If (s_n) is not bounded above, then $\limsup s_n = +\infty$. And by Thm 11.2 (ii) there exist a subsequence of (s_n) that diverges to $+\infty$.

Suppose (s_n) is bounded above, $\limsup\limits_{n\to\infty} s_n = t \in \mathbb{R}$

By Thm 11.2 (i) there exists a monotonic subsequence of (s_n) that converges to t iff . Fix $\varepsilon > 0$

$\limsup\limits_{n\to\infty} s_n = t \Rightarrow$

Suppose that . Then $\exists N_1 > N$ s.t.

Subsequential limits and convergence

Thm 11.8. Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n). Then

(i)

(ii)

(iii)

Proof. (iii) follows from (ii) and Thm 10.7

(ii) Suppose $t \in S \iff$

Then by Thm 10.7

Note that $\forall k$, therefore

and

Examples

For each sequence below let S denote the set of subsequential limits.

- $a_n = (-1)^n$,

① $S =$

$\lim_{k\to\infty} a_{2k-1} = -1$, $\lim_{k\to\infty} a_{2k} = 1$

If $t \notin \{-1, 1\}$, then

② $\limsup a_n =$ $\liminf a_n =$

- $b_n = 2^{n(-1)^n}$

① $S =$

$\lim_{k\to\infty} b_{2k-1} = 0$, $\lim_{k\to\infty} b_{2k} = +\infty$

If $t \in \mathbb{R}$, $t \neq 0$, then

② $\limsup b_n =$, $\liminf b_n =$

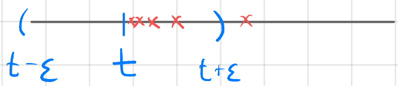
The set of subsequential limits is closes

Thm 11.9 Let (s_n) be a sequence. Denote by S the set of all subsequential limits of (s_n). Then

Let (t_n) be a sequence in $S \cap \mathbb{R}$, i.e. $\forall n \ (t_n \in S \cap \mathbb{R})$.
If (t_n) has a limit, then $\lim_{n\to\infty} t_n \in S$.

Proof Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Then

Fix $\varepsilon > 0$. Then



Since

and

$t_{n_0} \in S$ (subsequential limit) $\overset{\text{Thm 11.2}}{\Rightarrow}$

limsup's and liminf's

Thm 12.1 Let (s_n) and (t_n) be two sequences. Then

$$\left((s_n) \text{ converges} \wedge \lim_{n\to\infty} s_n = s > 0 \right) \Rightarrow \limsup_{n\to\infty} s_n t_n = s \cdot \limsup_{n\to\infty} t_n$$

Convention: For any $s \in \mathbb{R}$, $s > 0$, $s \cdot (+\infty) = +\infty$, $s \cdot (-\infty) = -\infty$.

Proof

① : $\limsup(s_n t_n) \geq s \cdot t$ (only for $\limsup t_n = t \in \mathbb{R}$)

Thm. 11.7 $\Rightarrow$ $\exists (t_{n_k})$ such that $\lim_{k\to\infty} t_{n_k} = t$. | Thm 9.4 $\Rightarrow$ $\lim_{k\to\infty}(s_{n_k} t_{n_k}) = s \cdot t$

Thm 11.3 $\Rightarrow$ $\lim_{k\to\infty} s_{n_k} = s$.

$\Rightarrow$ $s \cdot t$ is a $\Rightarrow$

② : $\limsup(s_n t_n) \leq s \cdot t$ (only for $s_n > 0 \ \forall n$). Thm 9.5 $\Rightarrow$

$\overset{①}{\geq}$

Then

①②

$\Rightarrow$

Remark

If (s_n) and (t_n) are two sequences, and $\lim_{n\to\infty} s_n = 0$, then there is nothing we can say in general about $\limsup (s_n t_n)$.

- $s_n = \frac{1}{n}$, $t_n = n$ $\Rightarrow$ $\limsup \frac{1}{n} \cdot n =$

- $s_n = \frac{1}{n^2}$, $t_n = n$ $\Rightarrow$ $\limsup \frac{1}{n^2} \cdot n =$
- $s_n = \frac{1}{n}$, $t_n = n^2$ $\Rightarrow$ $\limsup \frac{1}{n} \cdot n^2 =$

Also it is important that one sequence converges.

- $s_n = (0, 1, 0, 1, 0, 1, \dots)$ $\quad \limsup s_n =$

  $t_n = (1, 0, 1, 0, 1, 0, \dots)$ $\quad \limsup t_n =$
- $s_n = (-1)^n$, $t_n = (-1)^{n+1}$ $\quad \limsup s_n = \limsup t_n =$, but