MATH 142A: Introduction to Analysis

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Today: Properties of continuous functions

> Q&A: February 10

Next: Ross § 19

Week 6:

- Homework 5 (due Sunday, February 14)
- Regrades of HW3 (Monday, February 8 - Wednesday, February 10)

The maximum-value theorem

<u>Def. 18.7</u> Let f be a function and let $A \subset \mathrm{dom}(f)$. f is called bounded on A if .

<u>Thm 18.1</u> Let f be a function, $[a,b] \subset \mathrm{dom}(f)$, f is continuous on $[a,b]$

Then (i)

(ii)

<u>Proof</u> (i) Suppose that f is not bounded on $[a,b]$

$\Rightarrow$

1. (x_n) is bounded
2. $\forall k \quad a \leq x_{n_k} \leq b$
3. $\tilde{x} \in [a,b]$, f cont. on $[a,b]$

The maximum-value theorem

Proof (ii) Denote $M :=$. By (i),

1. $M = \sup\{f(x) : x \in [a,b]\}$
2. $\forall n \; \left(M - \frac{1}{n} < f(x_n) \le M\right)$
3. $\forall n \; (x_n \in [a,b])$
4. $y_0 \in [a,b] \Rightarrow f$ is continuous at y_0

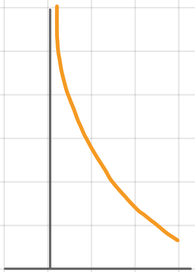
   $\Rightarrow$ by

(Exercise: prove that $\exists\, x_0 \in [a,b]$ s.t. $\forall x \in [a,b] \; (f(x_0) \le f(x))$)

Examples

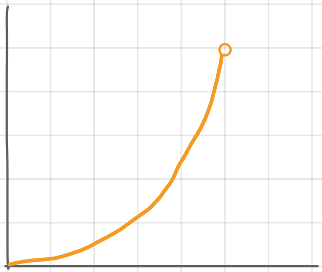
1) $f(x) = \frac{1}{x}$

- continuous on $(0,1]$
- unbounded on $(0,1]$



2) $f(x) = x^2$

- cont on $[0,1)$
- no maximum on $[0,1)$



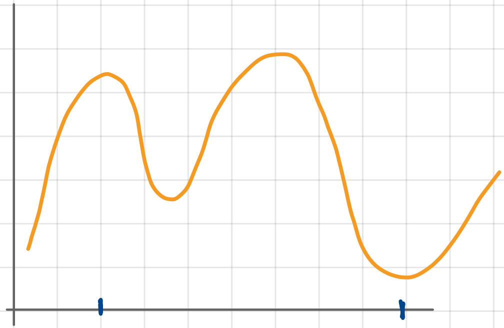
Intermediate value theorem

Thm 18.2 Let f be continuous on the interval $I \subset \mathbb{R}$. Let $a, b \in I$ s.t $a < b$. Then

(i) $f(a) < f(b)$ and $y \in (f(a), f(b))$

(ii) $f(a) > f(b)$ and $y \in (f(b), f(a))$

Proof (ii) Consider $S = \{x \in [a,b] : f(x) > y\}$



① $a \in S$, $\sup S \geq a$, $\sup(S) \leq \sup[a,b] = b$

$\Rightarrow$

②

Then

③ Define $t_n :=$

Image of an interval

Cor. 18.3 Let f be continuous on the interval I. Then

$$f(I) = \{ f(x) : x \in I \}$$

Proof If $\forall x \in I$ $f(x) = y_0$, then $f(I) = y_0$.

① Let $y_1 < y_2 \in f(I)$. Then

Let $y \in (y_1, y_2)$.

- If $x_1 < x_2$, then

$\Rightarrow$

- If $x_2 < x_1$, then

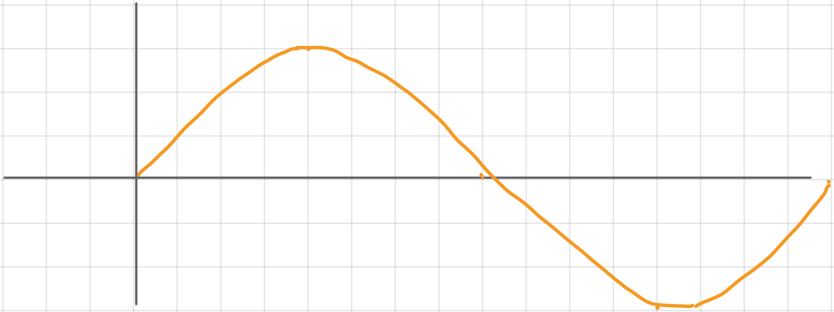
$\Rightarrow$

② Let $\inf f(I) < \sup f(I)$. Then $\forall y \in (\inf f(I), \sup f(I))$

Examples

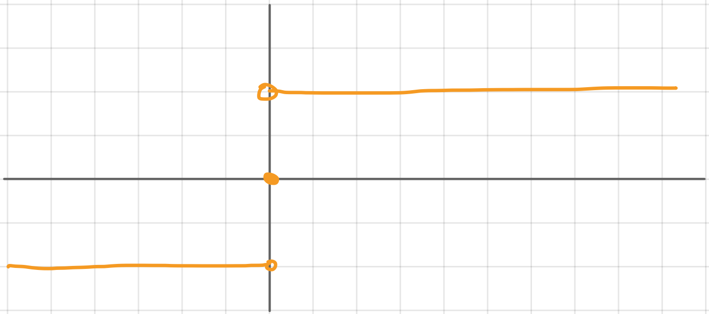
1) $\sin : (0, 2\pi) \to \mathbb{R}$

$$\sin((0, 2\pi)) \subset [-1, 1]$$



2) $f : [-1,1] \to \mathbb{R}$, $f(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

$f([-1,1]) =$



$$-1 = f\left(-\tfrac{1}{2}\right) < f(0) = 0$$

But $\forall \; y \in (-1, 0)$

$$\left\{ x \in \left(-\tfrac{1}{2}, 0\right) : f(x) = y \right\} = \emptyset$$

Continuity of strictly increasing functions.

Def 18.8 Function f is called

(strictly) increasing if $x<y \Rightarrow f(x) \le f(y)$ $\left(f(x) < f(y) \right)$

(strictly) decreasing if $x<y \Rightarrow f(x) \ge f(y)$ $\left(f(x) > f(y) \right)$

Thm 18.5 Let g be strictly increasing function on interval J.

If $g(J)$ is an interval, then

Proof. Let $x_0 \in J$, $x_0 > \inf J$, $x_0 < \sup J$. Then

Verify the ε-δ definition of continuity. Fix $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then

Now, $\forall x \in (x_1, x_2)$

Take Then

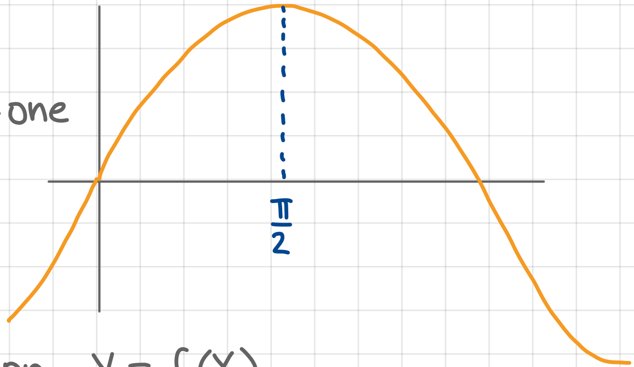
Inverse function

Def 18.9 Function $f: X \to Y$ is called **one-to-one (or bijection)**

if and

Example $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is one-to-one

$\sin : [0, \pi] \to [0,1]$ is not one-to-one



Def 18.10 Let $f: X \to Y$ be a bijection, $Y = f(X)$.

Then the function given by ( )

is called the **inverse of f**. In particular

Example
- $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$,
- $f: [0, +\infty) \to [0, +\infty)$, $f(x) = x^m$,

- If f is strictly increasing (decreasing) on X, then $f: X \to f(X)$ is a bijection

Continuity and the inverse function

<u>Thm 18.4</u> Let f be a continuous strictly increasing function on some interval I. Then $J := f(I)$ is an interval and

$f^{-1}: J \to I$ is

<u>Proof</u> ① f^{-1} is strictly increasing: Take $y_1, y_2 \in J$, $y_1 < y_2$

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then

If $x_1 \geq x_2$, then

② J is an interval: By Cor. 18.3 J is either an or

a . Since f is strictly increasing, J is an

③ ① + ② +

One-to-one continuous functions

Thm 18.6 Let f be a one-to-one continuous function on an interval I. Then f is or

Proof. ① If $a<b<c$ then either or

Otherwise, or

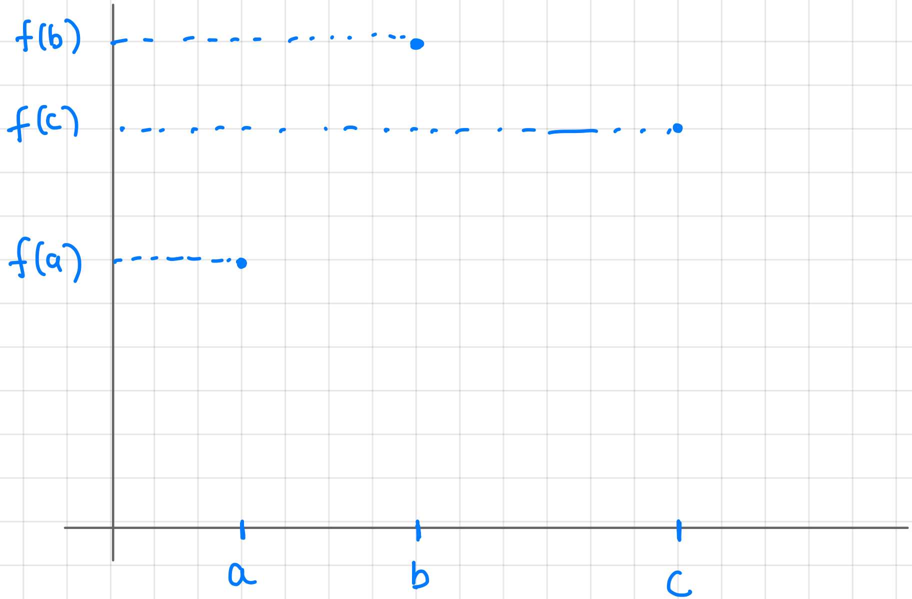
If $f(b)>\max\{f(a),f(c)\}$, choose

Then by Thm 18.2

. Similarly when $f(b)<\min\{f(a),f(c)\}$.

② Take any $a_0<b_0$. If $f(a_0)<f(b_0)$, then f is on I.

③ Similarly, if $f(a_0)>f(b_0)$, then f is decreasing.


f(b)
f(c)
f(a)
a
b
c