MATH 142A: Introduction to Analysis

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Today: Uniform continuity

> Q&A: February 12

Next: Ross § 19

Week 6:

- Homework 5 (due Sunday, February 14)

Inverse function

Def 18.9 Function $f: X \to Y$ is called **one-to-one (or bijection)**

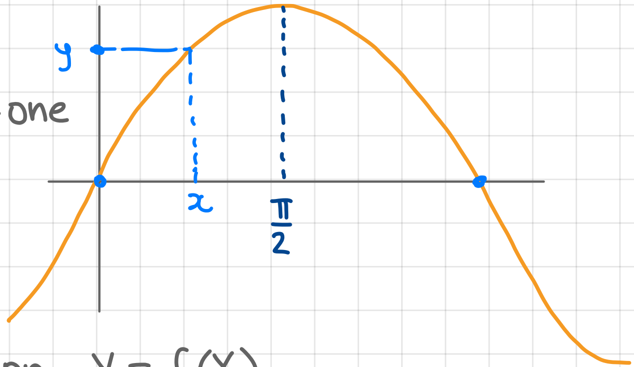
$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

if $f(X) = Y$ and $\forall y \in Y \ \exists!\, x \in X$ s.t. $f(x) = y$

Example $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is one-to-one

$\sin: [0, \pi] \to [0,1]$ is not one-to-one

$\sin(0) = \sin(\pi) = 0$



Def 18.10 Let $f: X \to Y$ be a bijection, $Y = f(X)$.

Then the function $f^{-1}: Y \to X$ given by $\left(f^{-1}(y) = x \Leftrightarrow f(x) = y \right)$ is called the **inverse of f**. In particular $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = y$

Example

- $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, $\sin^{-1} = \arcsin: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $f: [0,+\infty) \to [0,+\infty)$, $f(x) = x^m$, $f^{-1}: [0,+\infty) \to [0,+\infty)$, $f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$
- If f is strictly increasing (decreasing) on X, then $f: X \to f(X)$ is a bijection

Continuity and the inverse function

Thm 18.4 Let f be a continuous strictly increasing function on some interval I. Then $J := f(I)$ is an interval and

$f^{-1} : J \to I$ is continuous and strictly increasing.

Proof ① f^{-1} is strictly increasing: Take $y_1, y_2 \in J$, $y_1 < y_2$

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then $f(x_1) = y_1$, $f(x_2) = y_2$

If $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$, contradiction $\Rightarrow x_1 < x_2$

② J is an interval: By Cor. 18.3 J is either an interval or a single point. Since f is strictly increasing, J is an interval

③ ① + ② + Thm 18.5 f^{-1} is continuous on J.

One-to-one continuous functions

Thm 18.6 Let f be a one-to-one continuous function on an interval I. Then f is strictly increasing or strictly decreasing on I.

Proof. ① If $a<b<c$ then either $f(a)<f(b)<f(c)$ or $f(c)<f(b)<f(a)$

Otherwise, $f(b)>\max\{f(a),f(c)\}$ or $f(b)<\min\{f(a),f(c)\}$

If $f(b)>\max\{f(a),f(c)\}$, choose $y\in(\max\{f(a),f(c)\},f(b))$

Then by Thm 18.2 $\exists x_1\in(a,b)$ s.t. $f(x_1)=y$, $\exists x_2\in(b,c)$ s.t. $f(x_2)=y$

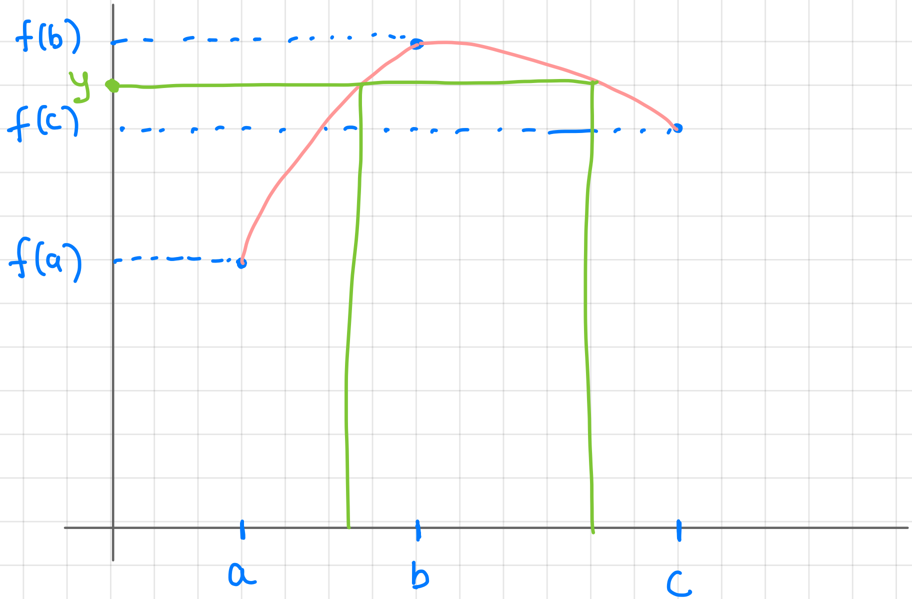
$\Rightarrow$ contradiction Similarly when $f(b)<\min\{f(a),f(c)\}$.

② Take any $a_0<b_0$. If $f(a_0)<f(b_0)$, then f is increasing on I.

$$x<a_0<y<b_0<z \overset{①}{\Rightarrow} f(a_0)<f(y)<f(b_0) \;\Bigg|\; \begin{array}{l} \Rightarrow f(x)<f(a_0)<f(y) \\ ① \\ \Rightarrow f(y)<f(b_0)<f(z) \end{array}$$

$$\Rightarrow \forall x_1<x_2 \quad (f(x_1)<f(x_2))$$

③ Similarly, if $f(a_0)>f(b_0)$, then f is decreasing. ∎



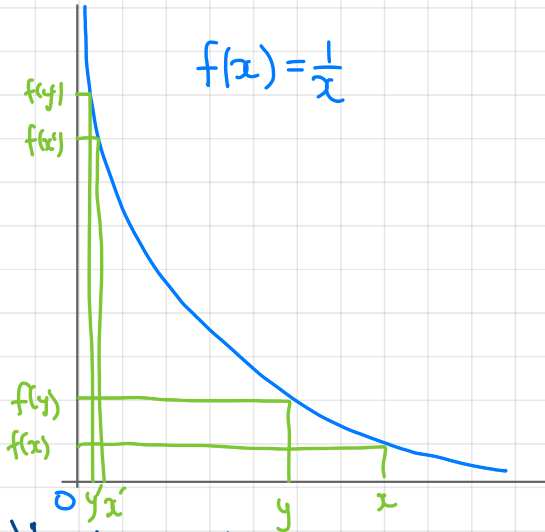
f(b)
y
f(c)
f(a)
a
b
c

Uniform continuity

Def. (Continuity on a set) Function f is continuous on $S \subset \mathbb{R}$ if $\forall x \in S \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall y \in S$ s.t. $|x-y| < \delta$ $\left(|f(x)-f(y)| < \varepsilon\right)$

Def. (Uniform continuity) Function f is uniformly continuous on $S \subset \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in S$ s.t. $|x-y| < \delta$ $\left(|f(x)-f(y)| < \varepsilon\right)$

Example Let $f(x) = \frac{1}{x}$.



1) $\forall [a,b] \subset (0, +\infty)$ f is unif. cont. on $[a,b]$.

Fix $\varepsilon > 0$. Then for $x, y \in [a,b]$

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2}$$

. Take $\delta = a^2 \cdot \varepsilon$

Then $|x-y| < \delta = a^2 \cdot \varepsilon \Rightarrow |f(x)-f(y)| < \frac{a^2 \cdot \varepsilon}{a^2} = \varepsilon$

2) f is not unif. cont. on $(0,1]$. Fix $\varepsilon = 1$

Then $\forall n$ $\left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)}$, but $\left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right| = |n+1-n| = 1$

Examples

3) $f(x) = x^2$ is continuous on $\mathbb{R}$, but is not unif. continuous on $\mathbb{R}$.

Take a sequence $x_n = \sqrt{n}$. Then

(i) $x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n}}$

$\forall\, \delta > 0 \;\; \exists\, n$ s.t. $|x_{n+1} - x_n| < \delta$

(ii) $|f(x_{n+1}) - f(x_n)| = |n+1-n| = 1$

$\Rightarrow f$ is not unif. cont. on $\mathbb{R}$

4) $f(x) = \cos(x^2)$ is continuous and bounded on $\mathbb{R}$, but not unif. continuous on $\mathbb{R}$

Take $x_n = \sqrt{\pi n}$. Then

(i) $\forall\, \delta > 0 \;\; \exists\, n$ s.t. $|x_{n+1} - x_n| < \delta$

(ii) $|f(x_{n+1}) - f(x_n)| = 2$

$\Rightarrow f$ is not unif. cont. on $\mathbb{R}$

Cantor - Heine Theorem

<u>Remark</u> If f is uniformly continuous on $S \subset \mathbb{R}$, then f is continuous on S.

<u>Thm 19.2</u> If f is continuous on a closed interval $[a,b]$, then f is uniformly continuous on $[a,b]$.

<u>Proof.</u> Suppose that f is cont. but not unif. cont. on $[a,b]$.

$$\Rightarrow \exists \varepsilon>0 \ \forall \delta>0 \ \exists x,y \in [a,b] \text{ s.t. } \left(|x-y|<\delta \wedge |f(x)-f(y)| \geq \varepsilon\right)$$

Take $\delta = \frac{1}{n}$: $\forall n \ \exists x_n, y_n \in [a,b]$ s.t. $\left(|x_n - y_n| < \frac{1}{n} \wedge |f(x_n) - f(y_n)| \geq \varepsilon\right)$

By the Bolzano - Weierstrass Thm 11.5 $\exists (x_{n_k}), (y_{n_k}), x_0, y_0 \in [a,b]$

$\lim x_{n_k} = x_0$, $\lim y_{n_k} = y_0$; since $|x_n - y_n| < \frac{1}{n}$, $\lim (x_{n_k} - y_{n_k}) = 0$

and thus $\lim x_{n_k} = \lim y_{n_k}$, $x_0 = y_0$. By continuity of f at x_0

$\lim f(x_{n_k}) = \lim f(y_{n_k}) = f(x_0)$, so $\lim \left(f(x_{n_k}) - f(y_{n_k})\right) = 0$, contradiction

$\blacksquare$

Uniform continuity

Thm 19.4 If f is uniformly continuous on a set S, and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence

Proof. Fix $\varepsilon>0$.

① f is unif. cont. on S

$$\Rightarrow \exists\, \delta>0 \text{ s.t. } \forall x,y\in S\ \left(|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\right)$$

② (s_n) is a Cauchy sequence

$$\Rightarrow \exists\, N\ \forall m,n>N\ \left(|s_n-s_m|<\delta\right)$$

$$\overset{①②}{\Longrightarrow} \forall\, m,n>N\ \ |f(s_n)-f(s_m)|<\varepsilon \Rightarrow (f(s_n)) \text{ is a Cauchy sequence}$$

∎

Example

Consider $f(x)=\frac{1}{x}$ and $t_n=\frac{1}{n}$. (t_n) is a Cauchy sequence, $\forall n\ t_n\in(0,1]$, but $f(t_n)=n$ is not a Cauchy sequence.

$\Rightarrow f$ is not unif. cont. on $(0,1]$.