MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Uniform continuity

> Q&A: February 12

Next: Ross § 19

Week 6:

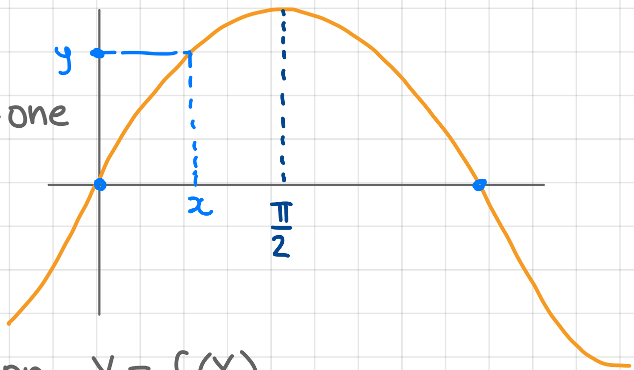
- Homework 5 (due Sunday, February 14)

Inverse function

Def 18.9 Function $f: X \to Y$ is called one-to-one (or bijection)

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

if $f(X) = Y$ and $\forall y \in Y \; \exists! \, x \in X$ s.t. $f(x) = y$



Example $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is one-to-one

$\sin : [0, \pi] \to [0,1]$ is not one-to-one

$\sin(0) = \sin(\pi) = 0$

Def 18.10 Let $f: X \to Y$ be a bijection, $Y = f(X)$.

Then the function $f^{-1}: Y \to X$ given by $\left(f^{-1}(y) = x \Leftrightarrow f(x) = y \right)$

is called the inverse of f. In particular $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = y$

Example

- $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, $\sin^{-1} = \arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $f: [0, +\infty) \to [0, +\infty)$, $f(x) = x^m$, $f^{-1}: [0, +\infty) \to [0, +\infty)$, $f^{-1}(x) = x^{\frac{1}{m}} = \sqrt[m]{x}$
- If f is strictly increasing (decreasing) on X, then $f: X \to f(X)$ is a bijection

Continuity and the inverse function

Thm 18.4 Let f be a continuous strictly increasing function on some interval I. Then $J := f(I)$ is an interval and

$f^{-1}: J \to I$ is

Proof ① f^{-1} is strictly increasing: Take $y_1, y_2 \in J$, $y_1 < y_2$

Denote $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then

If $x_1 \geq x_2$, then

② J is an interval: By Cor. 18.3 J is either an or

a Since f is strictly increasing, J is an

③ ① + ② +

One-to-one continuous functions

Thm 18.6 Let f be a one-to-one continuous function on an interval I. Then f is or

Proof. ① If $a<b<c$ then either or

Otherwise, or

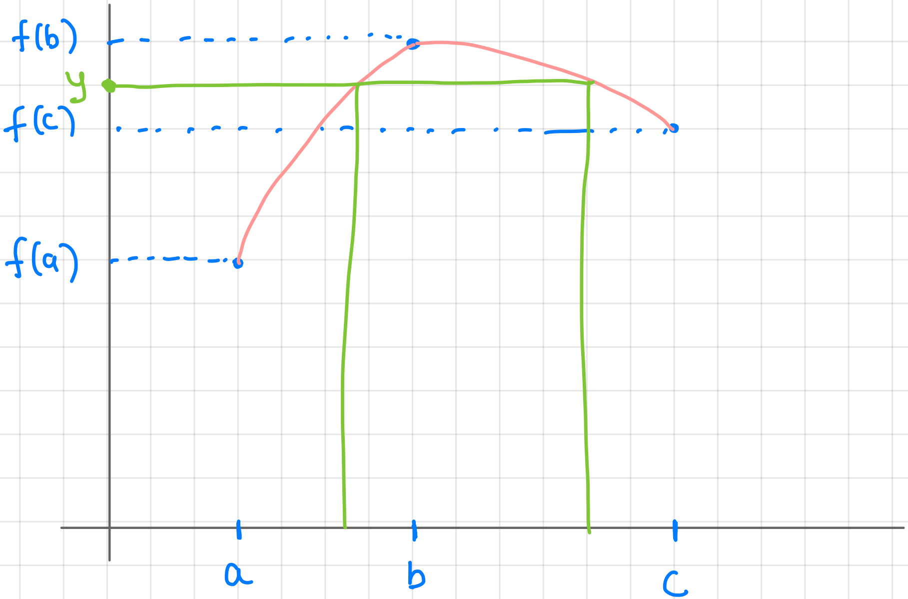
If $f(b)>\max\{f(a),f(c)\}$, choose

Then by Thm 18.2

Similarly when $f(b)<\min\{f(a),f(c)\}$.

② Take any $a_0<b_0$. If $f(a_0)<f(b_0)$, then f is on I.

③ Similarly, if $f(a_0)>f(b_0)$, then f is decreasing.

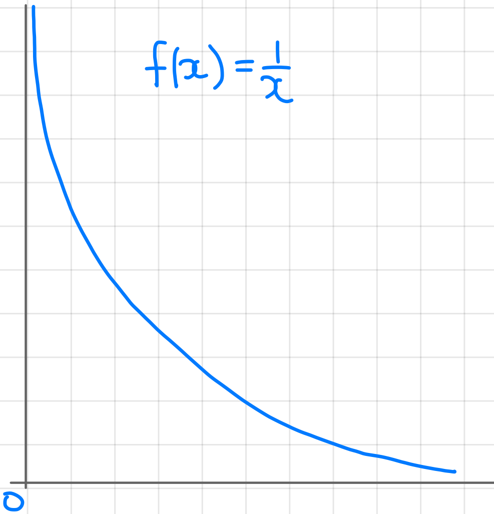

f(b)
y
f(c)
f(a)
a
b
c

Uniform continuity

Def. (Continuity on a set) Function f is continuous on $S \subset \mathbb{R}$ if $\forall x \in S$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall y \in S$ s.t. $|x - y| < \delta$ $(\,|f(x) - f(y)| < \varepsilon\,)$

Def. (Uniform continuity) Function f is uniformly continuous on $S \subset \mathbb{R}$ if

Example Let $f(x) = \frac{1}{x}$.



1) $\forall [a,b] \subset (0, +\infty)$ f is unif. cont. on $[a,b]$.

Fix $\varepsilon > 0$. Then for $x, y \in [a,b]$

. Take

Then

2) f is not unif. cont. on $(0,1]$. Fix

Then , but

Examples

3) $f(x) = x^2$ is continuous on $\mathbb{R}$, but is not unif. continuous on $\mathbb{R}$.

Take a sequence Then

(i)

(ii)

4) $f(x) =$ is continuous and bounded on $\mathbb{R}$, but not unif. continuous on $\mathbb{R}$

Take Then

(i)

(ii)

Cantor - Heine Theorem

Remark If f is uniformly continuous on $S \subset \mathbb{R}$, then f is continuous on S.

Thm 19.2 If f is continuous on a closed interval $[a,b]$, then f is

Proof. Suppose that f is cont. but not unif. cont. on $[a,b]$.

$\Rightarrow$

Take

and thus

, so

Uniform continuity

Thm 19.4 If f is uniformly continuous on a set S, and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence

Proof. Fix $\varepsilon > 0$.

① f is unif. cont. on S

② (s_n) is a Cauchy sequence

Example

Consider $f(x) = \frac{1}{x}$ and $t_n = \frac{1}{n}$. (t_n) is a Cauchy sequence, $\forall n$ $t_n \in (0,1]$, but $f(t_n) = n$ is not a Cauchy sequence.

$\Rightarrow$ f is not unif. cont. on $(0,1]$.