MATH 142A: Introduction to Analysis

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Today: Limits of functions

> Q&A: February 22

Next: Ross § 28

Week 8:

- Homework 7 (due Sunday, February 28)
- Midterm 2 (Wednesday, February 24): Lectures 8-16

Limit of a function, ε-δ definition

D 20.12 Let f be a functions defined on $S \subset \mathbb{R}$, let $a \in \mathbb{R}$ be a limit of some sequence in S, let $L \in \mathbb{R}$. We say that f tends to L as x tends to a along S if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in S \ \left(|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \right) \qquad (*)$$

Thm 20.6 Definitions 20.1 and 20.12 are equivalent.

Proof ($\Rightarrow$) Suppose that $(*)$ does not hold:

$$\exists \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists x_n \in S \ \left(|x_n - a| < \tfrac{1}{n} \ \wedge \ |f(x_n) - L| \geq \varepsilon \right)$$

$$\Rightarrow \exists (x_n) \text{ s.t. } \forall n \ x_n \in S, \ \lim x_n = a, \ \forall n \ |f(x_n) - L| \geq \varepsilon$$

contradiction to D 20.1

($\Leftarrow$) Let (x_n) be a sequence, $\forall n \ x_n \in S$, $\lim x_n = a$. [show $\lim f(x_n) = L$]

Fix $\varepsilon > 0$. Take δ as in $(*)$. $\lim x_n = a \Rightarrow \exists N \ \forall n > N \ |x_n - a| < \delta$

By $(*)$ $\forall n > N$ $|f(x_n) - L| < \varepsilon \Rightarrow \lim f(x_n) = L$

∎

Limit of a function, ε–δ definition

<u>D 20.13</u> Suppose that f is defined on $(a-c, a+c)\setminus\{a\}$ for some $c>0$. $a, L \in \mathbb{R}$

(a) We say that L is the (two-sided) limit of f at a if

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \left(0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon\right), \quad \lim_{x\to a} f(x) = L$$

(b) We say that L is the right-hand limit of f at a if

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \left(x \in (a, a+\delta) \Rightarrow |f(x)-L|<\varepsilon\right), \quad \lim_{x\to a^+} f(x) = L$$

(c) We say that L is the left-hand limit of f at a if

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \left(x \in (a-\delta, a) \Rightarrow |f(x)-L|<\varepsilon\right), \quad \lim_{x\to a^-} f(x) = L$$

<u>Corollary 20.7-20.8</u> Definitions 20.3 (a), (b), (c) and 20.13 (a), (b), (c) are equivalent.

<u>Proof</u> Follows from Thm 20.6 by specializing

(a) $S=(a-c, a+c)\setminus\{a\}$, (b) $S=(a, a+c)$, (c) $S=(a-c, a)$

Limit of a function

Suppose $f: S \to \mathbb{R}$, $a, L \in \mathbb{R}$

- $\lim_{x \to +\infty} f(x) = L \overset{\text{Def}}{\iff} \forall \varepsilon > 0 \ \exists t > 0 \ (x > t \Rightarrow |f(x) - L| < \varepsilon)$
- $\lim_{x \to +\infty} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists t > 0 \ (x > t \Rightarrow f(x) > M)$
- $\lim_{x \to +\infty} f(x) = -\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists t > 0 \ (x > t \Rightarrow f(x) < -M)$
- $\lim_{x \to -\infty} f(x) = L \overset{\text{Def}}{\iff} \forall \varepsilon > 0 \ \exists t > 0 \ (x < -t \Rightarrow |f(x) - L| < \varepsilon)$
- $\lim_{x \to -\infty} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists t > 0 \ (x < -t \Rightarrow f(x) > M)$
- $\lim_{x \to a} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists \delta > 0 \ (|x - a| < \delta \Rightarrow f(x) > M)$
- $\lim_{x \to a^-} f(x) = +\infty \overset{\text{Def}}{\iff} \forall M > 0 \ \exists \delta > 0 \ (x \in (a - \delta, a) \Rightarrow f(x) > M)$

Two-sided limits and left-hand/right-hand limits

Thm 20.10

Let f be a function defined on $J\setminus\{a\}$ for some open interval J containing $a \in \mathbb{R}$. Let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L \ \wedge \ \lim_{x \to a^-} f(x) = L$$

<u>Proof</u>. $(\Rightarrow)$ Exercise

$(\Leftarrow)$ Suppose $L \in \mathbb{R}$. Fix $\varepsilon > 0$.

$$\left.\begin{array}{l} \exists\, \delta_1 > 0 \ (x \in (a, a+\delta_1) \Rightarrow |f(x) - L| < \varepsilon) \\ \exists\, \delta_2 > 0 \ (x \in (a-\delta_2, a) \Rightarrow |f(x) - L| < \varepsilon) \end{array}\right| \Rightarrow \begin{array}{l} \delta = \min\{\delta_1, \delta_2\} \\ (0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon) \end{array}$$

Suppose $L = +\infty$. Fix $M > 0$.

$$\left.\begin{array}{l} \exists\, \delta_1 > 0 \ (x \in (a, a+\delta_1) \Rightarrow f(x) > M) \\ \exists\, \delta_2 > 0 \ (x \in (a-\delta_2, a) \Rightarrow f(x) > M) \end{array}\right| \Rightarrow \begin{array}{l} \delta = \min\{\delta_1, \delta_2\} \\ (0 < |x-a| < \delta \Rightarrow f(x) > M) \end{array}$$

Examples

1) $\lim_{x \to 0} \frac{\sin(7x)}{7x} = 1$

$g(y) = \begin{cases} \frac{\sin y}{y}, & y \neq 0 \\ 1, & y = 0 \end{cases}$ is continuous at 0, and defined on $\mathbb{R}$

$f(x) = 7x$, $\lim_{x \to 0} f(x) = 0$ $\Big/ \Rightarrow \lim_{x \to 0} g \circ f(x) = g(0) = 1$

2) Let $a > 1$, $p \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x^p}{a^x}$. Then

$\lim_{x \to +\infty} \frac{x^p}{a^x} = 0$

Fix $\varepsilon > 0$. By IE 6 $\lim_{n \to \infty} \frac{n^p}{a^n} = 0 \Rightarrow \lim \frac{(n+1)^p}{a^n} = 0$

$\Rightarrow \exists N \; \forall n > N \; \frac{(n+1)^p}{a^n} < \varepsilon$

Then $\forall x > N+1$ $[x] > N$ and $\left| \frac{x^p}{a^x} \right| = \frac{x^p}{a^x} \leq \frac{([x]+1)^p}{a^{[x]}} < \varepsilon$

Squeeze Lemma

Thm. 20.14 Let $f, g, h : S \to \mathbb{R}$, $\forall x \in S$ $f(x) \le g(x) \le h(x)$

Let $a, L \in \mathbb{R} \cup \{+\infty, -\infty\}$.

If $\lim_{S \ni x \to a} f(x) = \lim_{S \ni x \to a} h(x) = L$, then $\lim_{S \ni x \to a} g(x) = L$

Proof. Take any sequence (s_n) in S s.t. $\lim s_n = a$. Then

$\forall n$ $f(s_n) \le g(s_n) \le h(s_n)$, $\lim f(s_n) = \lim h(s_n) = L \overset{T9.11}{\Rightarrow} \lim g(x_n) = L$ ∎

IE 12 $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$. Fix $\varepsilon > 0$. By IE from Lecture 7,

$\lim \left(1 + \frac{1}{n}\right)^n = \lim \left(1 + \frac{1}{n}\right)^{n+1} = e$ and thus $\lim \left(1 + \frac{1}{n+1}\right)^n = \lim \left(1 + \frac{1}{n+1}\right)^{-1} \left(1 + \frac{1}{n+1}\right)^{n+1} = e$

$\Rightarrow \exists N_1$ $\forall n > N_1$ $\left| \left(1 + \frac{1}{n+1}\right)^n - e \right| < \varepsilon$, $\exists N_2$ $\forall n > N_2$ $\left| \left(1 + \frac{1}{n}\right)^{n+1} - e \right| < \varepsilon$

$\forall x > \max\{N_1, N_2\} + 1$

$$-\varepsilon < \left(1 + \frac{1}{[x]+1}\right)^{[x]} - e \le \left(1 + \frac{1}{x}\right)^x - e \le \left(1 + \frac{1}{[x]}\right)^{[x]+1} - e < \varepsilon$$ ∎