MATH 142A: Introduction to Analysis

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Today: Basic properties of the derivative

> Q&A: February 26

Next: Ross § 29

Week 8:

- Homework 7 (due Sunday, February 28)

Limits of functions. Examples

Warm up Last time: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

① $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{-n} =$

② $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-(n-1)} = \lim_{n\to\infty}\left(1-\frac{1}{n-1}\right)^{-n} =$

③ $\forall\ x<0$ $\quad \left(1+\frac{1}{\lceil x\rceil}\right)^{\lceil x\rceil+1} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{\lceil x\rceil+1}\right)^{\lceil x\rceil}$

④ Fix $\varepsilon>0$. $\exists N\ \forall n>N$ $\left|\left(1-\frac{1}{n}\right)^{-(n+1)} - e\right|<\varepsilon$, $\left|\left(1-\frac{1}{n+1}\right)^{-n} - e\right|<\varepsilon$

For $x<$

$< \left(1+\frac{1}{x}\right)^x - e <$

Important examples (limits of functions)

IE 13 $\quad \displaystyle\lim_{x\to 0} \frac{\log(1+x)}{x}$

Proof. ① $\dfrac{\log(1+x)}{x}$ is well-defined on $(-1,+\infty)\setminus\{0\}$

② Write $\dfrac{\log(1+x)}{x} =$

③ $\displaystyle\lim_{x\to 0_+} (1+x)^{\frac{1}{x}} =$: Let (x_n) be a sequence in $(0,1)$

$\lim x_n = 0$. Define Then

and

④ $\displaystyle\lim_{x\to 0_-} (1+x)^{\frac{1}{x}} =$. As in ③

⑤ By Thm 20.10

⑥ log is continuous on $(0,+\infty) \Rightarrow$

Important examples (limits of functions)

IE 14

$$\lim_{x \to 0} \frac{e^x - 1}{x} =$$

Proof Denote $f(x) :=$, so that $x =$

Then $\frac{e^x - 1}{x} =$ where

① $f(x)$ is continuous on $\mathbb{R}$,

② g is defined on $(-1, +\infty)$, by

$\Rightarrow$

Important examples (limits of functions)

IE 15

$\forall \alpha \in \mathbb{R}$

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} =$$

Proof ① Write $\frac{(1+x)^\alpha - 1}{x} =$

② Denote $f(x) =$

$g(y) =$

Then by IE 14 , so

by Thm 20.5

③ By IE 13

Differentiability and derivative

<u>Def</u> Let $f: I \to \mathbb{R}$, I open interval. Let $a \in I$.
We say that f is differentiable at $a \in I$, or that f has a derivative at a, if the limit

exists and is finite. If f is differentiabe $\forall a \in I$, we get a function

<u>Examples</u> 1) Let $f(x) = x$. Then $\forall a \in \mathbb{R}$ $f'(a) = 1$ (so $f'(x) = 1$)

2) Let $f(x) = \sin x$. Then $f'(x) = \cos x$

Examples

3) $(e^x)' = e^x$

For any $x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

Thm 28.2 f is differentiable at point a $\Rightarrow$ f is continuous at a

Proof. f differentiable at a $\Rightarrow$ $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

Rewrite $f(x) =$

Then $\lim\limits_{x \to a} f(x) =$

Derivatives and arithmetic operations

Thm 28.3 Let f and g be differentiable at a, $c\in\mathbb{R}$. Then $c\cdot f$, $f+g$ and $f\cdot g$ are differentiable at a. If additionally $g(a)\neq 0$, then $\frac{f}{g}$ is differentiable at a. Moreover

$$(cf)'(a) = c\cdot f'(a),\quad (f+g)'(a) = f'(a)+g'(a),\quad (f\cdot g)'(a) = f'(a)g(a)+f(a)g'(a)$$

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$$

Proof. $(cf)'$, $(f+g)'$ – exercise.

$$\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim_{x\to a}\frac{\big(f(x)-f(a)\big)g(x)+f(a)\big(g(x)-g(a)\big)}{x-a}$$

$$= f'(a)\cdot g(a) + f(a)\cdot g'(a)$$

If $g(a)\neq 0$, then

$$\lim_{x\to a}\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a}\frac{1}{g(x)g(a)}\cdot\frac{f(x)g(a)-f(a)g(x)-f(a)g(a)+f(a)g(a)}{x-a} = \frac{f'(a)g(a)-f(a)g'(a)}{\big(g(a)\big)^2}$$

$\blacksquare$

Derivative of a composition

Thm 28.4 If f is differentiable at a, and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) =$$

Remark

$$\frac{g(f(x)) - g(f(a))}{x-a} =$$

Take $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$, $g(y) = e^y$: $\lim\limits_{x \to 0} \dfrac{e^{x^2 \sin \frac{1}{x}} - e^0}{x^2 \sin\left(\frac{1}{x}\right)}$ is not well defined $\left(x_n = \frac{1}{\pi n}\right)$

Proof: ① g is defined on $(f(a)-c, f(a)+c)$ for some $c > 0$.

f is cont. at $a \Rightarrow$

Derivative of a composition

Case 1: $\exists\ \eta \le \delta$ s.t $\forall\ x \in (a-\eta, a+\eta)$

Then $\dfrac{g(f(x)) - g(f(a))}{f(x) - f(a)}$ can be written on $(a-\eta, a+\eta)$ as $\varphi \circ f(x)$

where $\varphi(y) = \Big\{$ is defined on $(f(a)-c, f(a)+c)$

g is differentiable at $f(a) \Rightarrow$

By Thm 20.5

$$\Rightarrow \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} = \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$=$$

Case 2: $\exists (x_n)$, $\lim x_n = a$, $\forall n \ x_n \neq a$, $f(x_n) = f(a)$

T20.2 $\rightarrow$ f is continuous at a, and

$\hookrightarrow$ g is continuous at $f(a)$, $f(x_n) = f(a)$ $\forall n$, so

if $\lim\limits_{x \to a} \dfrac{g\circ f(x) - g\circ f(a)}{x-a} = (g\circ f)'(a)$ exists, then

$$(g\circ f)'(a) =$$

Fix $\varepsilon > 0$.

Then

Then $\forall x \in (a-\delta', a+\delta') \setminus \{a\}$, $f(x) \neq f(a)$