MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Mean Value Theorem

> Q&A: March 1

Next: Ross § 30

Week 8:

- Homework 8 (due Sunday, March 7)

Fermat's Theorem

Thm 29.1 (i) $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$
(ii) f assumes its max or min at x_0
(iii) $f'(x_0)$ exists
$\Rightarrow f'(x_0)=0$

Proof. Suppose that f assumes its max at x_0 (otherwise take $-f$)

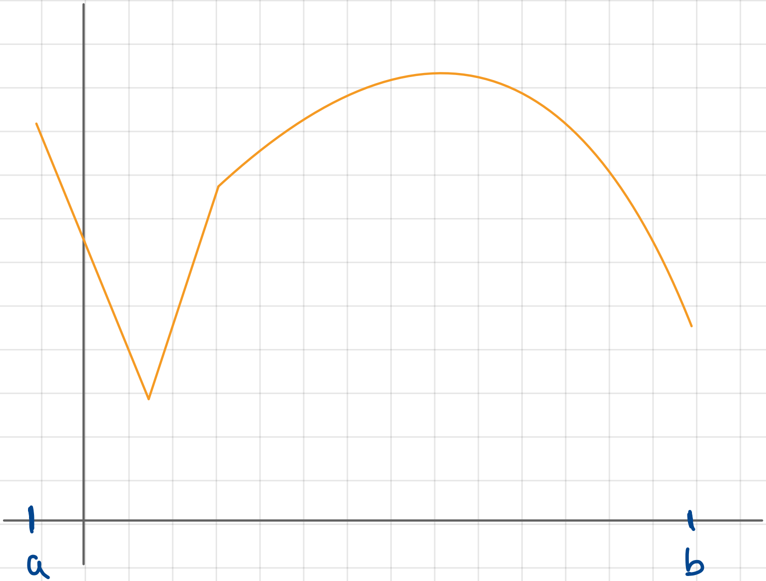
If $\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=f'(x_0)>0$, then $\exists\,\delta>0$ $\forall x\in(x_0-\delta,x_0+\delta)$

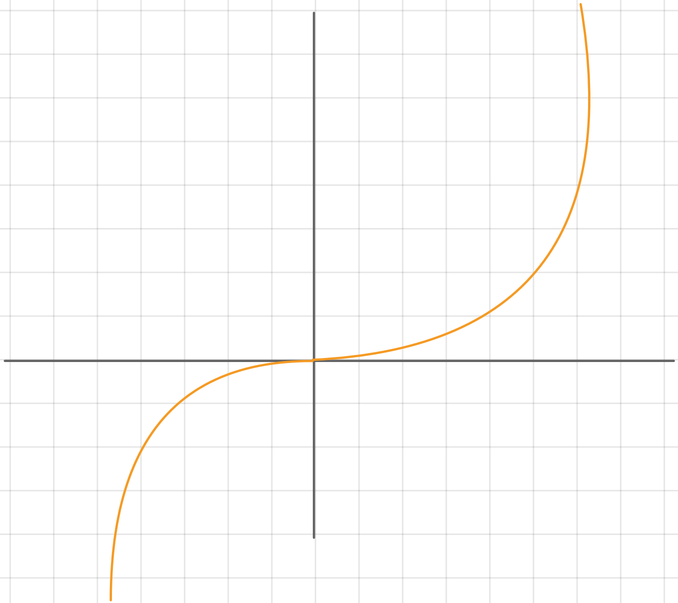
$$\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|<\frac{f'(x_0)}{2}\quad=\quad\frac{f(x)-f(x_0)}{x-x_0}>\frac{f'(x_0)}{2}>0,$$

so $\forall x\in(x_0,x_0+\delta)$ $\quad f(x)-f(x_0)>0 \Leftrightarrow f(x)>f(x_0)$ contradiction

Therefore, $f'(x_0)\leq 0$. Similar argument shows that $f'(x_0)\geq 0$ ∎

Critical points





Rolle's Theorem

Notation: If $S \subset \mathbb{R}$ then

- $f \in C(S)$ means that f is continuous on S
- $f \in D(S)$ means that f is differentiable on S

Thm 29.2

(i) $f \in C([a,b])$

(ii) $f \in D((a,b))$ $\Rightarrow \exists\, c \in (a,b)$ s.t. $f'(c) = 0$

(iii) $f(a) = f(b)$

Proof. By the maximum-value theorem (Thm 18.1)

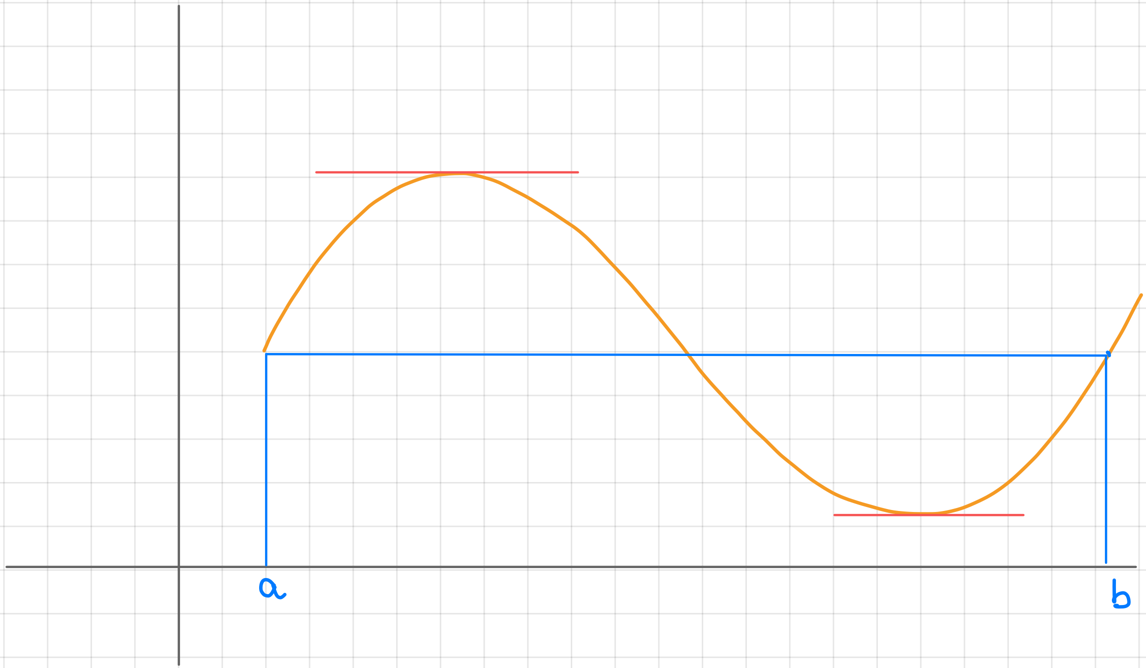
$$\exists\, x_0, y_0 \in [a,b] \text{ s.t. } \forall x \in [a,b] \quad f(x_0) \le f(x) \le f(y_0)$$

If $\{x_0, y_0\} = \{a, b\}$, then $f(x_0) = f(y_0) \Rightarrow \forall x \in [a,b]$ $f(x) = f(a)$, $f'(x) = 0$

If $y_0 \in (a,b)$, then by Thm 29.1 $f'(y_0) = 0$

If $x_0 \in (a,b)$, then by Thm 29.1 $f'(x_0) = 0$ ∎

Rolle's Theorem



Mean-value Theorem (Lagrange's Theorem)

Thm 29.3

(i) $f \in C([a,b])$
(ii) $f \in D((a,b))$

$\Rightarrow \exists c \in (a,b)$ s.t $f(b) - f(a) = f'(c)(b-a)$

Proof. Denote $F : [a,b] \to \mathbb{R}$, $F(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$

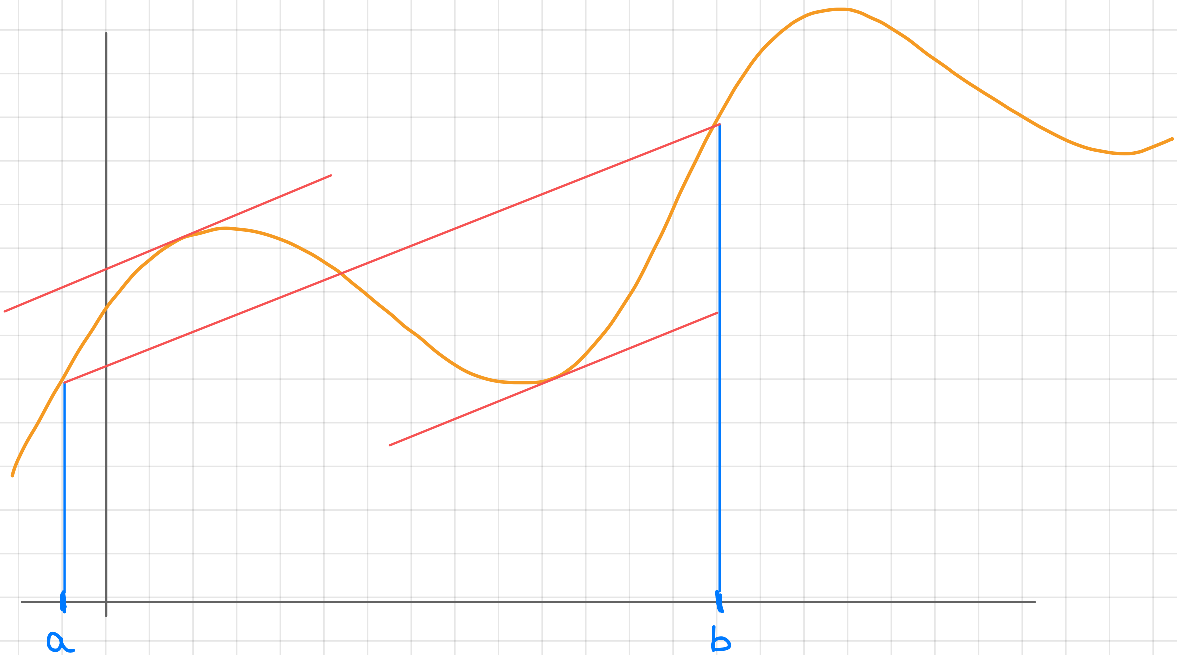
Then $F \in C([a,b])$
$F \in D((a,b))$
$F(a) = F(b) = f(a)$

$\xRightarrow{\text{Rolle's Thm}}$ $\exists\, c \in (a,b)$ s.t. $F'(c) = 0$

Since $F'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$, we get $f(b) - f(a) = f'(c)(b-a)$

$\blacksquare$

Mean-value Theorem (Lagrange's Theorem)



Corollaries

Cor. 29.4 (i) $f \in D((a,b))$
(ii) $f' = 0$ on (a,b) $\Big| \Rightarrow \exists\, C \in \mathbb{R}$ s.t. $\forall x \in (a,b)$ $f(x) = C$

Proof (By contradiction). If $\exists\, x, y \in (a,b)$ s.t. $f(x) \neq f(y)$, then by Lagrange's Thm $\exists\, c \in (x,y)$ s.t. $f'(c) = \dfrac{f(y) - f(x)}{\underset{\neq 0}{y - x}}$

■

Cor 29.5 (i) $f, g \in D((a,b))$
(ii) $f' = g'$ on (a,b) $\Big| \Rightarrow \exists\, C \in \mathbb{R}$ s.t. $f = g + C$ on (a,b)

Proof Apply Cor. 29.4 to $f - g$:

■

Application of Thms 29.1-29.3

1) $\forall\ x, y \in \mathbb{R} \quad |\sin x - \sin y| \le |x - y|$

Fix $x, y \in \mathbb{R}$, $x < y$. $\sin \in C([x,y])$, $\sin \in D((x,y))$, so by Lagrange's thm

$\exists\ c \in (x,y)$ s.t. $\sin y - \sin x = \sin'(c)\,(y-x)$ and thus

$$|\sin y - \sin x| = |\cos c|\,|y-x| \le |y-x|$$

2) $\forall\ x, y \in [1, +\infty) \quad |\sqrt{x} - \sqrt{y}| \le \frac{1}{2}|x-y|$

Fix $x, y \in [1, +\infty)$, $x < y$. Let $f : [0, +\infty) \to [0, +\infty)$, $f(u) = \sqrt{u}$. Then

$f \in C([x,y])$, $f \in D((x,y))$, so by Lagrange's Thm

$\exists\ c \in (x,y)$ s.t. $f(y) - f(x) = f'(c)\,(y-x)$, $f'(c) = \frac{1}{2\sqrt{c}}$, and thus

$$|f(y) - f(x)| = \frac{1}{2\sqrt{c}}\,|y-x| \le \frac{1}{2}\,|y-x|$$

$\blacksquare$

Application of Thms 29.1-29.3

3) $\forall x \in \mathbb{R}$ $\quad e^x \geq 1+x$, equality only at $x=0$

Let $x>0$, $f(u)=e^u$. $f \in C([0,x])$, $f \in D((0,x))$, $f'(u)=e^u$, so by Lagrange's thm $\exists\, c \in (0,x)$ s.t. $f(x)-f(0)=e^c(x-0)>x$

(since $e^c > e^0 = 1$)

If $x<0$, apply Lagrange's thm to $f \in C([x,0])$, $f \in D((x,0))$.

Then $\exists\, c \in (x,0)$ s.t. $f(0)-f(x) = e^c(0-x) < -x$

Therefore, $\forall\, x \neq 0$ $\quad e^x > 1+x$

Monotonic functions and the mean-value theorem

<u>Def</u>. 29.6 Let $I \subset \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$. We say that

- f is strictly increasing on I if $\forall\, x, y \in I$ $(x<y \Rightarrow f(x)<f(y))$
- f is strictly decreasing on I if $\forall\, x, y \in I$ $(x<y \Rightarrow f(x)>f(y))$
- f is increasing on I if $\forall\, x, y \in I$ $(x<y \Rightarrow f(x)\leq f(y))$
- f is decreasing on I if $\forall\, x, y \in I$ $(x<y \Rightarrow f(x)\geq f(y))$

<u>Cor 29.7</u>. $f \in D((a,b))$. Then

(i) f is strictly increasing on (a,b) if $f'(x)>0$ for all $x \in (a,b)$

(ii) f is strictly decreasing on (a,b) if $f'(x)<0$ for all $x \in (a,b)$

(iii) f is increasing on (a,b) if $f'(x)\geq 0$ for all $x \in (a,b)$

(iv) f is decreasing on (a,b) if $f'(x)\leq 0$ for all $x \in (a,b)$

<u>Proof</u>. (ii) Take $x, y \in (a,b)$, $x<y$. By Lagrange's thm $\exists\, c \in (x,y)$ s.t.

$$f(y)-f(x) = f'(c)(y-x) < 0 \qquad \blacksquare$$

Intermediate-value theorem for derivatives (Darboux's Thm)

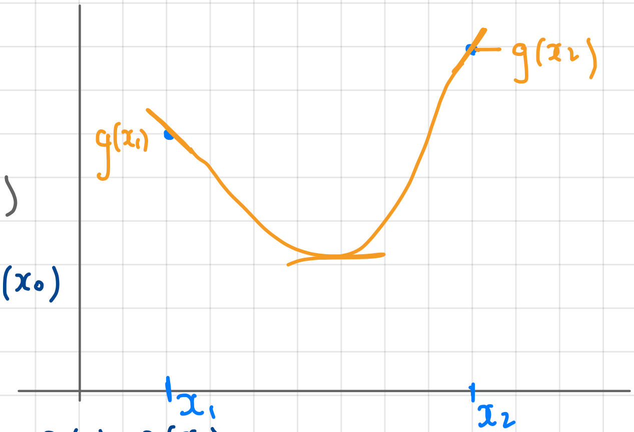
Thm 29.8 $f \in D((a,b))$, $x_1, x_2 \in (a,b)$, $x_1 < x_2$.

(i) $f'(x_1) < f'(x_2) \Rightarrow \forall c \in (f'(x_1), f'(x_2))\ \exists x \in (x_1, x_2)$ s.t. $f'(x) = c$

(ii) $f'(x_1) > f'(x_2) \Rightarrow \forall c \in (f'(x_2), f'(x_1))\ \exists x \in (x_1, x_2)$ s.t. $f'(x) = c$

Proof: (i) Fix $c \in (f'(x_1), f'(x_2))$.

Consider $g(x) = f(x) - cx$. Then



① $g \in C([x_1, x_2])$, by Thm 18.1 (max-value)

$\exists x_0 \in [x_1, x_2]$ s.t. $\forall x \in [x_1, x_2]$ $g(x) \geq g(x_0)$

② $g'(x_1) < 0 < g'(x_2)$

$\lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} < 0 \Rightarrow \exists \delta > 0\ \forall x \in (x_1, x_1 + \delta)$ $\frac{g(x) - g(x_1)}{x - x_1} < 0 \Rightarrow x_0 \neq x_1$

Similarly, $x_0 \neq x_2$. So $x_0 \in (x_1, x_2)$

③ $g \in D((x_1, x_2)) \xRightarrow{\text{Fermat's Thm}} g'(x_0) = 0 \Rightarrow g'(x_0) = f'(x_0) - c = 0 \Rightarrow f'(x_0) = c$. ∎