# MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Derivative of the inverse.
L'Hôpital's rule
> Q&A: March 1

Next: Ross § 31

- Homework 8 (due Sunday, March 7)
- CAPE at www.cape.ucsd.edu

## Derivative of the inverse

$f: I \to J$, $f^{-1}: J \to I$, $\forall x \in I$ $f^{-1} \circ f(x) = x$, $\forall y \in J$ $f \circ f^{-1}(y) = y$

If $f \in D(I)$, $f^{-1} \in D(J)$, then differentiating both sides gives

$\forall x \in I$ $(f^{-1} \circ f)'(x) = 1$, $\forall y \in J$ $(f \circ f^{-1})'(y) = 1$

By the chain rule

$$(f \circ f^{-1})'(y) = f'(f^{-1}(y)) \, (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \qquad (*)$$

If $f^{-1}$ exists and $f$ and $f^{-1}$ are differentiable, then $(f^{-1})'$ is given by $(*)$.

Suppose $f: I \to J$, $f^{-1}: J \to I$ exists and $f$ is differentiable at $x_0 \in I$. Does this imply that $f^{-1}$ is differentiable at $y_0 = f(x_0)$?

# Derivative of the inverse

**Thm. 29.9.** Let $f: I \to J$ be one-to-one and continuous on $I$.

(i) $f$ is differentiable at $x_0$

(ii) $f'(x_0) \neq 0$ $\Rightarrow$

**Proof.** Fix $\varepsilon > 0$.

① $f'(x_0) \neq 0 \Leftrightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \neq 0$

Consider $g := f^{-1}$, $g: J \to I$.

② Thms 18.6, 18.4 $\Rightarrow$

③ $\forall\ y \in (y_0 - \eta, y_0 + \eta) \setminus \{y_0\}$

## Examples

1. $\arcsin = \sin^{-1}$,

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-1,1)$ is a bijection (strictly increasing)

$\forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \sin'(x) =$

Let $y \in (-1,1)$ and let $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ s.t. $\sin x = y$

by Thm 29.9 arcsin is diffentiable at $y$ and

$\arcsin'(y) =$

2. $\log : (0,+\infty) \to \mathbb{R}$ is the inverse of $x \mapsto e^x$

$e^x \in D(\mathbb{R})$, $(e^x)' = e^x$, $e^x > 0$

$\Rightarrow \forall\, y \in (0,+\infty)$ log is differentiable at $y$

and $(\log y)' =$

## Examples

3. $f : \mathbb{R} \to (0, +\infty)$ , $f(x) = a^x$ $(a > 0, a \neq 1)$

$f(x) =$ $\Rightarrow \forall x \in \mathbb{R}$ $f'(x) \overset{T28.4}{=}$

4. $\log_a : (0, +\infty) \to \mathbb{R}$ is the inverse of $x \mapsto a^x$ , $\forall x \in \mathbb{R}$ $a^x > 0$,

So and

$$(\log_a y)' \overset{T29.9}{=}$$

# L'Hôpital's rule

Consider the limit $\lim\limits_{S \ni x \to a} \frac{f(x)}{g(x)}$, $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $S \subset \mathbb{R}$

- if $\lim\limits_{S \ni x \to a} f(x) =: F \in \mathbb{R}$, $\lim\limits_{S \ni x \to a} g(x) =: G \in \mathbb{R} \setminus \{0\}$, then

$$\lim_{S \ni x \to a} \frac{f(x)}{g(x)} \overset{T.20.4}{=} \frac{F}{G}$$

- if $F \in \{+\infty, -\infty\}$ and $G \in \{+\infty, -\infty\}$ $\quad \frac{\infty}{\infty}$ | usual tools don't work

  $F = 0$ and $G = 0$ $\quad \frac{0}{0}$

$f, g$ differentiable $\Rightarrow$ try L'Hôpital's rule

# Generalized mean value theorem (Cauchy's Thm)

**Thm 30.1** $f, g \in C([a,b])$, $f, g \in D((a,b))$ $\Rightarrow$ $\exists\, x \in (a,b)$ s.t.

**Proof** Consider $h(x) :=$

$h \in C([a,b])$

$h \in D((a,b))$

$h(a) =$

$h(b) =$

If $g(b) \neq g(a)$, $g'(x) \neq 0$, then

# L'Hôpital's Rule

**Thm 30.2** Let $a \in \mathbb{R}$ and $s$ signify $a, a^+, a^-, +\infty$ or $-\infty$.

Suppose that $f$ and $g$ are differentiable (on appropriately chosen intervals) and that $\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$ exists.

Then if

(i) $\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$

OR

(ii) $\lim_{x \to s} |g(x)| = \infty$

$\Rightarrow$

**Proof** Only for $s = a^-$ and for $s = +\infty$ (other cases: exercise)

# Proof of L'Hôpital's rule

① Suppose $-\infty < L \leq +\infty$. Take $L_1 < L$.

$\lim\limits_{x \to s} \frac{f'(x)}{g'(x)}$ exists $\Rightarrow$

By Darboux's thm. either
Cor 29.7

$\Rightarrow$

$\Rightarrow$

Take $K \in (L_1, L)$. $\lim\limits_{x \to s} \frac{f'(x)}{g'(x)} = L > K \Rightarrow$

By Cauchy's thm $\forall [x,y] \subset (\alpha, s)$ $\exists z \in (x,y)$ s.t.

$$(f(y)-f(x))g'(z) = (g(y)-g(x))f'(z) \Rightarrow$$

If (i) holds, take

# Proof of L'Hôpital's rule

If (ii) holds, then $\exists\ \alpha_1 \in (\alpha, s)$ s.t. $\forall\ [x,y] \subset (\alpha_1, s)$

$$\Rightarrow \forall\ [x,y] \subset (\alpha_1, s) \qquad \frac{f(y)-f(x)}{g(y)-g(x)} \cdot \frac{g(y)-g(x)}{g(y)}$$

$$\Rightarrow \frac{f(y)}{g(y)} =$$

Take the limit (for any fixed $x \in (\alpha_1, s)$)

$$\lim_{y \to s} \frac{f(x)-Kg(x)}{g(y)} = \qquad \Rightarrow \exists\ \alpha_2 \in (\alpha_1, s) \text{ s.t. } \forall \quad y \in (\alpha_2, s)$$

$$\frac{f(x)-Kg(x)}{g(y)} >$$

Conclusion:

# Proof of L'Hôpital's rule

(2) If $-\infty \le L < +\infty$, then

(3) Suppose $L \in \mathbb{R}$. Fix $\varepsilon > 0$. Take $L_1 = L - \varepsilon$, $L_2 = L + \varepsilon$

(A) $\Rightarrow$

(B) $\Rightarrow$

$\Rightarrow$

Suppose $L = +\infty$. Fix $M > 0$. Take $L_1 = M$.

(A) $\Rightarrow$

Suppose $L = -\infty$. Fix $M > 0$. Take $L_2 = -M$ $\Rightarrow \lim\limits_{x \to s} \frac{f(x)}{g(x)} = -\infty$

# Examples

1. For any $\alpha > 0$

$$\lim_{x \to +\infty} \frac{\log x}{x^{\alpha}}$$

2. $\forall \; a > 1$ and $0 < \alpha < n$

$$\lim_{x \to +\infty} \frac{x^{\alpha}}{a^{x}}$$

3. $\lim_{x \to 0} \frac{\sin x}{x} =$