MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Higher-order derivatives
Taylor's formula
> Q&A: March 5
Next: Ross § 31

- Homework 8 (due Sunday, March 7)
- CAPE at www.cape.ucsd.edu

Higher-order derivatives

$f: I \to \mathbb{R}$, $f \in D(I)$, $f': I \to \mathbb{R}$

If $f' \in D(I)$, we get a new function $(f')': I \to \mathbb{R}$, called the second derivative of f, denoted $f''(x)$, $\dfrac{d^2 f(x)}{dx^2}$

Def. 31.14 By induction, if the derivative $f^{(n-1)}(x)$ of order $n-1$ of f has been defined, then the derivative of order n is defined by $f^{(n)}(x) = \left(f^{(n-1)}\right)'(x)$. Denoted $f^{(n)}(x)$, $\dfrac{d^n f(x)}{dx^n}$

If f has derivative of order n on I, we write $f \in D^{(n)}(I)$

Examples

$f(x)$	$f'(x)$	$f''(x)$	$f^{(n)}(x)$
a^x	$a^x \log a$	$a^x (\log a)^2$	$a^x (\log a)^n$
x^α	$\alpha x^{\alpha-1}$	$\alpha(\alpha-1) x^{\alpha-2}$	$\alpha(\alpha-1)\cdots(\alpha-n+1) x^{\alpha-n}$
$\log x$	x^{-1}	$(-1) x^{-2}$	$(-1)^{n-1}(n-1)!\, x^{-n}$

Examples

Example 1 (Leibniz' formula) Let $f, g \in D^{(n)}(I)$, $n \in \mathbb{N}$.

Then $$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x),$$ where $\binom{n}{k} = \frac{n!}{k!\,(n-k}$

Proof (Exercise) By induction: $n=1$ follows from Thm 28.3

Induction step: suppose $(f \cdot g)^{(n-1)} = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)} \cdot g^{(n-1-k)}$

Then $$(f \cdot g)^{(n)}(x) = \left(\sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)} \cdot g^{(n-1-k)} \right)' = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(f^{(k+1)} g^{(n-1-k)} + f^{(k)} g^{(n-k)} \right) = \cdots$$ $\blacksquare$

Example 2 Consider $P_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, $c_k \in \mathbb{R}$, $k \in \{0, \dots, n\}$

$P_n(0) = c_0$; $P_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^n \Rightarrow P_n'(0) = c_1$

$P_n''(x) = 2c_2 + 3 \cdot 2 \cdot c_3 \cdot x + \cdots + n \cdot (n-1)\, x^{n-2} \Rightarrow P_n''(0) = 2c_2$

$P_n^{(3)}(x) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot x + \cdots + n(n-1)(n-2)\, c_n x^{n-3} \Rightarrow P_n^{(3)}(0) = 3!\, c_3$

$\forall k \in \{0, \dots, n\}$ $P_n^{(k)}(0) = k!\, c_k \Rightarrow P_n(x) = P_n(0) + \frac{P_n^{(1)}(0)}{1!} x + \frac{P_n^{(2)}(0)}{2!} x^2 + \cdots + \frac{P_n^{(n)}(0)}{n!} x^n$

Taylor's formula

Let $x_0 \in \mathbb{R}$. Consider polynomial

$$P_n(x_0; x) = c_0 + c_1(x - x_0) + c_2(x-x_0)^2 + \cdots + c_n(x-x_0)^n$$

Then

$$P_n(x_0; x) = P_n(x_0; x_0) + \frac{P'(x_0; x_0)}{1!}(x - x_0) + \frac{P''(x_0; x_0)}{2!}(x-x_0)^2 + \cdots + \frac{P^{(n)}(x_0; x_0)}{n!}(x - x_0)^n$$

Def. 31.15 Let $f: I \to \mathbb{R}$, f has derivatives up to order n at $x_0 \in I$. Then we call the polynomial

$$P_n(x_0; x) := f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

the Taylor polynomial of order n of $f(x)$ at x_0. We call the function $R_n(x_0; x) := f(x) - P_n(x_0; x)$ the n-th remainder in Taylors formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x_0; x)$$

Taylor's Theorem

Thm 31.16 Let $x, x_0 \in \mathbb{R}$, let $I(\bar{I})$ be open (closed) interval with end points x and x_0. Let

$$f \in D^{(n)}(\bar{I}),\ f \in D^{(n+1)}(I),\ f, f', f'', \dots, f^{(n)} \in C(\bar{I})$$

Then for any function $\varphi \in C(\bar{I}), \varphi \in D(I)$, $\forall x \in I\ \varphi'(x) \neq 0$, there exists $\xi \in I$ s.t.

$$R_n(x_0; x) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi)\, n!} \cdot f^{(n+1)}(\xi)(x-\xi)^n$$

Cor 31.17 (Cauchy's form of the remainder term)

If we take $\varphi(t) = x - t$, $\varphi'(x) = -1$ and $R_n(x_0; x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-x_0)$

Cor 31.3 (Lagrange's form of the remainder term)

If we take $\varphi(t) = (x-t)^{n+1}$ $\varphi'(t) = -(n+1)(x-t)^n$, $R_n(x_0; x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

Taylor's Theorem

Proof. Consider function $F(t) = f(x) - P_n(t; x)$

$$F(t) = f(x) - \left[f(t) + \frac{f'(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right] \Rightarrow F \in C(\bar{I}),\ F \in D(I)$$

By Cauchy's theorem $\exists\, \xi \in I$ s.t. $\dfrac{\cancel{F(x)}^{\,0} - F(x_0)}{\varphi(x) - \varphi(x_0)} = \dfrac{F'(\xi)}{\varphi'(\xi)}$

$F(x) = 0$, $F(x_0) = R_n(x_0; x)$

$$\Rightarrow R_n(x_0; x) = -\frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi)} \cdot F'(\xi)$$

$$F'(t) = -\left[\cancel{f'(t)} - \cancel{\frac{f'(t)}{1!}} + \cancel{\frac{f''(t)}{1!}}(x-t) - \cancel{\frac{f''(t)}{1!}}(x-t) + \cancel{\frac{f^{(3)}(t)}{2!}}(x-t)^2 - \cancel{\cdots} + \cdots \right.$$

$$\left. - \cancel{\frac{f^{(k)}(t)}{(k-1)!}}(x-t)^{k-1} + \cancel{\frac{f^{(k+1)}(t)}{k!}}(x-t)^k - \cdots + \cdots - \cancel{\frac{f^{(n)}(t)}{(n-1)!}}(x-t)^{n-1} + \boxed{\frac{f^{(n+1)}(t)}{n!}(x-t)^n} \right]$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

$$\left(\frac{f^{(k)}(t)}{k!}(x-t)^k \right)' = -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k$$

■

Examples

IE 16 Take $f(x)=e^x$, $x\in\mathbb{R}$. Then for $x_0=0$ Taylor's formula gives

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + R_n(0;x)$$

with the remainder (Lagrange's form)

$$R_n(0;x) = \frac{1}{(n+1)!}e^{\xi}\cdot x^{n+1}, \text{ where } |\xi|<|x|$$

Thus

$$|R_n(0;x)| = \frac{1}{(n+1)!}e^{\xi}|x|^{n+1} < \frac{|x|^{n+1}}{(n+1)!}e^{|x|}$$

For any $x\in\mathbb{R}$

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0 \quad (\text{IE } 7), \text{ so } \lim_{n\to\infty} R_n(0;x) = 0$$

$$-R_n(0;x) = \sum_{k=0}^{n}\frac{x^k}{k!} - e^x \quad \Rightarrow \forall x\in\mathbb{R} \quad \sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$$

In particular, $e = \sum_{k=0}^{\infty}\frac{1}{k!}$ $\quad (0!=1)$

Examples

IE 17 Take $f(x)=\sin(x)$, $x\in\mathbb{R}$. Then $f^{(n)}(x)=\sin(x+\frac{\pi}{2}n)$, and the remainder in Lagrange's form for $x_0=0$ is

$$|R_n(0;x)|=\left|\frac{1}{(n+1)!}\sin\left(\xi+\frac{\pi(n+1)}{2}\right)x^{n+1}\right|\leq\frac{|x|^{n+1}}{(n+1)!}\to 0,\ n\to\infty$$

Therefore, $\forall x\in\mathbb{R}$ $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

$$\sin^{(n)}(0)=\sin\left(\frac{\pi n}{2}\right)=\begin{cases}0, & n=2k\\ 1, & n=4k+1\\ -1, & n=4k-1\end{cases}$$

Similarly,

$$\forall x\in\mathbb{R}\quad \cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\dots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Examples

IE 18 Take $f(x) = \log(1+x)$, $x \in (-1,1]$. $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Then the remainder in Lagrange's form for $x_0 = 0$ is

$$R_n(0;x) = \frac{(-1)^n n!\, x^{n+1}}{(n+1)!\,(1+\xi)^{n+1}} = \frac{(-1)^n}{n+1}\left(\frac{x}{1+\xi}\right)^{n+1}$$

If $x \in (0,1]$, $\xi \in (0,x)$, $0 < \dfrac{x}{1+\xi} < x \le 1$, so $R_n(0,x) \to 0$, $n \to \infty$

If $x \in (-1,0)$, $\xi \in (x,0)$, $\left|\dfrac{x \;{\scriptstyle -0.9}}{1+\xi \;{\scriptstyle -0.7}}\right|$ is not necessarily less than 1

Remainder in Cauchy's form gives

$$R_n(0;x) = \frac{(-1)^n \cancel{n!}\,(x-\xi)^n x}{\cancel{n!}\,(1+\xi)^{n+1}} = \left(\frac{\xi - x}{1+\xi}\right)^n \frac{x}{1+\xi}$$

$$0 < \frac{\xi - x}{1+\xi} = 1 - \frac{1+x}{1+\xi} < 1 - \frac{1+x}{1} = -x < 1 \implies R_n(0;x) \to 0,\ n \to \infty$$

$$\implies \forall x \in (-1,1] \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$$