MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Higher-order derivatives

Taylor's formula

> Q&A: March 5

Next: Ross § 31

- Homework 8 (due Sunday, March 7)
- CAPE at www.cape.ucsd.edu

Higher-order derivatives

$f: I \to \mathbb{R}$, $f \in D(I)$, $f': I \to \mathbb{R}$

If $f' \in D(I)$, we get a new function $(f')': I \to \mathbb{R}$, called the second derivative of f, denoted

Def. 31.14 By induction, if the derivative $f^{(n-1)}(x)$ of order $n-1$ of f has been defined, then the derivative of order n is defined by . Denoted

If f has derivative of order n on I, we write

Examples

$f(x)$	$f'(x)$	$f''(x)$	$f^{(n)}(x)$
a^x	$a^x \log a$		
x^α	$\alpha x^{\alpha-1}$		
$\log x$	x^{-1}		

Examples

Example 1 (Leibniz' formula) Let $f, g \in D^{(n)}(I)$, $n \in \mathbb{N}$.

Then $(f \cdot g)^{(n)}(x) =$, where $\binom{n}{k} =$

Proof. (Exercise) By induction: $n=1$ follows from Thm 28.3

Induction step: suppose $(f \cdot g)^{(n-1)} = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)} \cdot g^{(n-1-k)}$

Then $(f \cdot g)^{(n)}(x) = \left(\sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)} \cdot g^{(n-1-k)} \right)' = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(f^{(k+1)} g^{(n-1-k)} + f^{(k)} g^{(n-k)} \right) = \cdots$ ∎

Example 2 Consider $P_n(x) =$, $c_k \in \mathbb{R}$, $k \in \{0, \ldots, n\}$

$P_n(0) =$; $P_n'(x) =$ $\Rightarrow$

$P_n''(x) =$ $\Rightarrow$

$P_n^{(3)}(x) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot x + \cdots + n(n-1)(n-2) c_n x^{n-3} \Rightarrow P_n^{(3)}(0) = 3! c_3$

$\forall k \in \{0, \ldots, n\}$ $P_n^{(k)}(0) =$ $\Rightarrow P_n(x) = P_n(0) + \frac{P_n^{(1)}(0)}{1!} x + \frac{P_n^{(2)}(0)}{2!} x^2 + \cdots + \frac{P_n^{(n)}(0)}{n!} x^n$

Taylor's formula

Let $x_0 \in \mathbb{R}$. Consider polynomial

$$P_n(x_0; x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \dots + c_n(x - x_0)^n$$

Then

$$P_n(x_0; x) = P_n(x_0; x_0) + \frac{P'(x_0; x_0)}{1!}(x - x_0) + \frac{P''(x_0; x_0)}{2!}(x - x_0)^2 + \dots + \frac{P^{(n)}(x_0; x_0)}{n!}(x - x_0)^n$$

Def. 31.15 Let $f: I \to \mathbb{R}$, f has derivatives up to order n at $x_0 \in I$. Then we call the polynomial

the Taylor polynomial of order n of $f(x)$ at x_0. We call the function the n-th remainder in Taylors formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x_0; x)$$

Taylor's Theorem

Thm 31.16 Let $x, x_0 \in \mathbb{R}$, let $I(\bar{I})$ be open (closed) interval with end points x and x_0. Let

Then for any function , $\forall x \in I$

there exists s.t.

$$R_n(x_0;x) =$$

Cor 31.17 (Cauchy's form of the remainder term)

If we take $\varphi(t) =$, $\varphi'(x) = -1$ and $R_n(x_0;x) =$

Cor 31.3 (Lagrange's form of the remainder term)

If we take $\varphi(t) =$ $\varphi'(\xi) =$, $R_n(x_0;x) =$

Taylor's Theorem

Proof. Consider function $F(t) =$

$$F(t) = f(x) - \left[f(t) + \frac{f'(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right] \Rightarrow$$

By Cauchy's theorem $\exists\, \xi \in I$ s.t.

$F(x) = 0$, $F(x_0) =$

$$\Rightarrow R_n(x_0; x) =$$

$$F'(t) = -\left[f'(t) - \frac{f'(t)}{1!} + \frac{f''(t)}{1!}(x-t) - \frac{f''(t)}{1!}(x-t) + \frac{f^{(3)}(t)}{2!}(x-t)^2 - \cdots + \cdots \right.$$

$$\left. - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \cdots + \cdots - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n \right]$$

$$=$$

$$\left(\frac{f^{(k)}(t)}{k!}(x-t)^k \right)' = -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k$$

Examples

IE 16 Take $f(x) =$, $x \in \mathbb{R}$. Then for $x_0 =$ Taylor's formula gives

$$e^x =$$

with the remainder (Lagrange's form)

$$R_n(0;x) = \qquad\qquad , \text{ where}$$

Thus

$$|R_n(0;x)| =$$

For any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \qquad\qquad (\text{IE } 7), \text{ so } \lim_{n\to\infty} R_n(0;x) =$$

$$-R_n(0;x) = \sum_{k=0}^{n} \frac{x^k}{k!} - e^x \qquad \Rightarrow \forall x \in \mathbb{R}$$

In particular, $e =$ $\qquad\qquad (0! = 1)$

Examples

IE 17 Take $f(x)=\sin(x)$, $x\in\mathbb{R}$. Then $f^{(n)}(x)=\sin(x+\frac{\pi}{2}n)$, and the remainder in Lagrange's form for $x_0=0$ is

$$|R_n(0;x)|=\left|\frac{1}{(n+1)!}\sin\left(\xi+\frac{\pi(n+1)}{2}\right)x^{n+1}\right|\leq\frac{|x|^{n+1}}{(n+1)!}\to 0,\ n\to\infty$$

Therefore, $\forall x\in\mathbb{R}$ $\sin x =$

$$\sin^{(n)}(0)=\sin\left(\frac{\pi n}{2}\right)=\begin{cases}0, & n=2k\\ 1, & n=4k+1\\ -1, & n=4k-1\end{cases}$$

Similarly,

$\forall x\in\mathbb{R}$ $\cos(x)=$

Examples

IE 18 Take $f(x) =$ $\quad$, $x \in (-1, 1]$. $f^{(n)}(x) =$

Then the remainder in Lagrange's form for $x_0 = 0$ is

$$R_n(0; x) =$$

If $x \in (0, 1]$, $\xi \in (0, x)$, $0 < \frac{x}{1+\xi} < x \leq 1$, so $R_n(0, x) \to 0$, $n \to \infty$

If $x \in (-1, 0)$, $\xi \in (x, 0)$, $\left|\frac{x}{1+\xi}\right|$ is not necessarily less than 1

Remainder in Cauchy's form gives

$$R_n(0; x) =$$

$0 < \frac{\xi - x}{1+\xi} =$ $\qquad$ $< 1 \Rightarrow R_n(0; x) \to 0$, $n \to \infty$

$\Rightarrow \forall x \in (-1, 1]$ $\log(1+x) =$ $\qquad$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} =$