

MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Taylor's formula
Little-o/big-O notation
> Q&A: March 8

Next: -

Week 10:

- Homework 9 (due Sunday, March 14)
- Quiz 5 (Wednesday, March 10, Lectures 17-20)
- Regrades for HW 6 and HW 7 (until Friday, March 12)
- CAPE at www.cape.ucsd.edu

Taylor's formula

Let $f: I \rightarrow \mathbb{R}$, f has derivatives up to order n at $x_0 \in I$.

Taylor's formula:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \frac{f^{(2)}(x_0)}{2!} (x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + R_n(x_0; x)$$

Taylor's Thm: If $f \in D^{(n)}(\bar{I})$, $f \in D^{(n+1)}(I)$, $f, f', f^{(2)}, \dots, f^{(n)} \in C(\bar{I})$.

then for any function $\varphi \in C(\bar{I})$, $\varphi \in D(I)$, $\forall x \in I$ $\varphi'(x) \neq 0$

there exists $\xi \in I$ s.t.

$$R_n(x_0; x) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi) n!} f^{(n+1)}(\xi) (x-\xi)^n$$

Cauchy's form of the remainder term $R_n(x_0; x) = \frac{f^{(n+1)}(\xi)}{n!} (x-\xi)^n (x-x_0)$

Lagrange's form of the remainder term $R_n(x_0; x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}$

Example

IE 19 Let $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$, $x > -1$. Then (Lecture 22)

$$\forall n \in \mathbb{N} \quad f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$

Taylor's formula at $x_0 = 0$:

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + R_n(0;x)$$

Cauchy's form of the remainder (ξ between x and 0)

$$R_n(0;x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)\cdot(1+\xi)^{\alpha-n-1}}{n!} \cdot (x-\xi)^n x = \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\xi)^{\alpha-1}}{n!} \left(\frac{x-\xi}{1+\xi}\right)^n x$$

For $|x| < 1$ $\left| \frac{x-\xi}{1+\xi} \right| = \frac{|x| - |\xi|}{|1+\xi|} \leq \frac{|x| - |\xi|}{1 - |\xi|} = 1 - \frac{1-|x|}{1-|\xi|} \leq 1 - \frac{1-|x|}{1} = |x|$, so

$$|R_n(0;x)| \leq (1+|x|)^{\alpha-1} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} |x|^{n+1} =: C_n ; \quad \frac{C_{n+1}}{C_n} = \left| \frac{n+1}{\alpha-(n+1)} \right| |x| \rightarrow |x| \text{ as } n \rightarrow \infty$$

$$\Rightarrow \exists \rho \in (x, 1) \exists N \in \mathbb{N} \forall k \in \mathbb{N} \quad \frac{C_{n+k}}{C_n} < \rho^k \Rightarrow \lim_{n \rightarrow \infty} C_n = 0 \Rightarrow \lim_{n \rightarrow \infty} R_n(0;x) = 0$$

$\alpha = n \in \mathbb{N} \Rightarrow$ Newton binomial Thm ; if $\alpha = -1 \Rightarrow$ geometric series

Taylor series. Analytic functions

Def 3.1.8. If the function $f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at x_0 , we call the series

$$f(x_0) + \frac{1}{1!} f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + \dots$$

the **Taylor series** of f at point x_0 .

Remarks 1) If f has derivatives of all orders at x_0 , this does not imply that the Taylor series of f at x_0 converges

2) If the Taylor series of f at x_0 converges, then this

does not imply that $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n = f(x)$ (*)

Functions that satisfy (*) are called **analytic**

Example of a non-analytic function $f(x) = \begin{cases} 0, & x=0 \\ e^{-\frac{1}{x^2}}, & x \neq 0 \end{cases}$

$$f^{(n)}(0) = 0 \quad \forall n = 0, 1, 2, \dots \text{ (exercise)}$$

Comparison of the Asymptotic Behavior of functions

Def 31.19 • Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \rightarrow \mathbb{R}$, $c < s$, we say that f is infinitesimal compared with g as x tends to s , and write $f = o(g)$ as $x \rightarrow s$

if there exist $c' \geq c$ and $h : (c', s) \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) \cdot h(x) \quad \text{on } (c', s) \quad \text{and} \quad \lim_{x \rightarrow s} h(x) = 0$$

• Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \rightarrow \mathbb{R}$, $c > s$ we say that f is infinitesimal compared with g as x tends to s , and write $f = o(g)$ as $x \rightarrow s$, if there exist $c' \leq c$ and $h : (s, c') \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) \cdot h(x) \quad \text{on } (s, c') \quad \text{and} \quad \lim_{x \rightarrow s} h(x) = 0$$

• $f = o(g)$ as $x \rightarrow a$ if $f = o(g)$ as $x \rightarrow a^+$ and $f = o(g)$ as $x \rightarrow a^-$

Comparison of the Asymptotic Behavior of functions

Def 3.19 • Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \rightarrow \mathbb{R}$, $c < s$, we write $f = O(g)$ as $x \rightarrow s$

if there exist $c' \geq c$ and $B : (c', s) \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) B(x) \quad \text{on } (c', s) \quad \text{and } B \text{ is bounded on } (c', s)$$

• Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \rightarrow \mathbb{R}$, $c > s$ we write $f = O(g)$ as $x \rightarrow s$, if there exist $c' \leq c$, $B : (s, c') \rightarrow \mathbb{R}$

s.t. $f(x) = g(x) \cdot B(x)$ on (s, c') and B is bounded on (s, c')

• $f = O(g)$ as $x \rightarrow a$ if $f = O(g)$ as $x \rightarrow a^+$ and $f = O(g)$ as $x \rightarrow a^-$

• We say that f and g are of the same order as $x \rightarrow s$

and write $f \asymp g$ as $x \rightarrow s$ if $f = O(g)$ and $g = O(f)$ as $x \rightarrow s$

$\Leftrightarrow \exists c_1, c_2 \in (0, +\infty)$ s.t. $c_1 |g(x)| \leq |f(x)| \leq c_2 |g(x)|$ on the corresponding interval

Comparison of the Asymptotic Behavior of functions

Def 3.1.9 • Let $a \in \mathbb{R}$ and $s \in \{a^-, +\infty\}$. For $f, g : (c, s) \rightarrow \mathbb{R}$, $c < s$, we say that f is equivalent to g as x tends to s , and write $f \sim g$ as $x \rightarrow s$, if there exist $c' \geq c$ and $\gamma : (c', s) \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) \cdot \gamma(x) \quad \text{on } (c', s) \quad \text{and} \quad \lim_{x \rightarrow s} \gamma(x) = 1$$

• Let $a \in \mathbb{R}$ and $s \in \{a^+, -\infty\}$. For $f, g : (s, c) \rightarrow \mathbb{R}$, $c > s$ we say that f is equivalent to g as x tends to s , and write $f \sim g$ as $x \rightarrow s$, if there exist $c' \leq c$ and $\gamma : (s, c') \rightarrow \mathbb{R}$ such that

$$f(x) = g(x) \cdot \gamma(x) \quad \text{on } (s, c') \quad \text{and} \quad \lim_{x \rightarrow s} \gamma(x) = 1$$

• $f \sim g$ as $x \rightarrow a$ if $f \sim g$ as $x \rightarrow a^+$ and $f \sim g$ as $x \rightarrow a^-$

Examples

1) $x^2 = \overset{f}{x} \cdot \overset{g}{x} \overset{h}{x} \Rightarrow x^2 = o(x)$ as $x \rightarrow 0$

2) $x = \overset{f}{\frac{1}{x}} \cdot \overset{h}{x^2} \overset{g}{x^2} \Rightarrow x = o(x^2)$ as $x \rightarrow +\infty$

3) $\frac{1}{x^2} = \frac{1}{x} \cdot \frac{1}{x}$ on $(0, +\infty) \Rightarrow \frac{1}{x^2} = o\left(\frac{1}{x}\right)$ as $x \rightarrow +\infty$

4) $\frac{1}{x} = x \cdot \frac{1}{x^2}$ on $(0, 1) \Rightarrow \frac{1}{x} = o\left(\frac{1}{x^2}\right)$ as $x \rightarrow 0^+$

5) For $a > 1$, $\lim_{x \rightarrow +\infty} \frac{x^n}{a^x} = 0$, $x^n = a^x \cdot \frac{x^n}{a^x}$ on $(0, +\infty) \Rightarrow x^n = o(a^x)$ as $x \rightarrow +\infty$

6) $\forall a > 0, a \neq 1, \forall \alpha > 0 \lim_{x \rightarrow +\infty} \frac{\log_a x}{x^\alpha} = 0 \Rightarrow \log_a x = o(x^\alpha)$ as $x \rightarrow +\infty$

7) $x = x \cdot 1 \Rightarrow x = o(1)$ as $x \rightarrow 0$

8) $\left(\frac{1}{x} + \sin x\right) \cdot x = O(x)$ as $x \rightarrow \infty$

9) $(2 + \sin x) \cdot x \asymp x$ as $x \rightarrow \infty$, but $(1 + \sin x)x$ is not of the same order as x as $x \rightarrow \infty$

10) $x^2 + x = x^2 \left(1 + \frac{1}{x}\right) \Rightarrow x^2 + x \sim x^2$ as $x \rightarrow \infty$

Taylor's formula

Lemma 31.20 Let $x_0 \in \mathbb{R}$, \bar{I} be a closed interval with endpoint x_0 , let φ be a function defined on \bar{I} , $\varphi \in D^{(n)}(\bar{I})$, and

$$\varphi(x_0) = \varphi'(x_0) = \dots = \varphi^{(n)}(x_0) = 0. \quad \text{Then}$$

$$\varphi(x) = o((x-x_0)^n) \quad \text{as } x \rightarrow x_0 \text{ along } \bar{I}. \quad (**)$$

Proof. (By induction). If $n=1$, then

$$\varphi(x) = \varphi(x_0) + \frac{\varphi(x) - \varphi(x_0)}{x - x_0} (x - x_0), \quad \varphi(x_0) = 0, \varphi'(x_0) = 0 \Rightarrow \varphi(x) = o(x - x_0) \text{ as } x \rightarrow x_0 \text{ along } \bar{I}$$

Suppose $(**)$ holds for $n=k-1$. Consider $\varphi' \in D^{(k-1)}(\bar{I})$, $\varphi'(x_0) = 0$

$$(\varphi')'(x) = (\varphi')''(x_0) = \dots = (\varphi')^{(k-1)}(x_0) = 0 \Rightarrow \varphi'(x) = o((x-x_0)^{k-1}) \quad \text{as } x \rightarrow x_0 \text{ along } \bar{I}$$

By Lagrange's thm, for $x \in \bar{I}$ close enough to x_0 $\exists \xi$ between x_0 and x

$$\varphi(x) - \varphi(x_0) = \varphi'(\xi)(x - x_0) = h(\xi)(\xi - x_0)^{k-1}(x - x_0), \quad h(x) \rightarrow 0 \text{ as } \bar{I} \ni x \rightarrow x_0$$

$$\Rightarrow |\varphi(x)| \leq |h(\xi)| |x - x_0|^k \Rightarrow \varphi(x) = o((x - x_0)^k), \quad \text{proves induction step. } \blacksquare$$

Taylor's formula (local). Peano's form of the remainder

Thm 31.21 Let $x_0 \in \mathbb{R}$, \bar{I} be a closed interval with endpoint x_0 ,

let f be a function defined on \bar{I} , $f \in D^{(n)}(\bar{I})$. Then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ + O((x-x_0)^n) \text{ as } x \rightarrow x_0, x \in \bar{I}$$

Proof. Apply Lemma 31.20 with $\varphi(x) = R_n(x_0; x)$ ■

Remark If $f \in D^{(n+1)}(I)$ and $f^{(n+1)}$ is bounded near x_0 , then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ + O((x-x_0)^{n+1}) \text{ as } x \rightarrow x_0, x \in \bar{I}$$

Examples

1) Asymptotic formulas as $x \rightarrow 0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + o(x^{n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n+1} x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + o(x^n)$$

2) Approximate \sin by a polynomial P_n s.t. $\max_{x \in [-1,1]} |\sin x - P_n(x)| \leq 10^{-3}$

Take $P_n = P_n(0; x)$ Taylor's polynomial at 0. By Lagrange's form

$$|R_{2n+2}(0; x)| = \left| \frac{\sin(\xi + \frac{\pi}{2}(2n+3))}{(2n+3)!} \right| |x|^{2n+3} \leq \frac{1}{(2n+3)!} < \frac{1}{1000} \text{ for } n=2$$
$$\Rightarrow \sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ on } [-1,1]$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{x + o(x^2)}{x}$$

$$= 1 + \lim_{x \rightarrow 0} \frac{o(x^2)}{x^2} \cdot x = 1$$