MATH 142A: Introduction to Analysis

www.math.ucsd.edu/~ynemish/teaching/142a

Today: Set of real numbers and completeness axiom

> Q&A: January 11

Next: Ross § 7

Week 2:

- Quiz 1 (Wednesday, January 13) - Lectures 1-2
- homework 1 (due Friday, January 15)

Maximum and minimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

Def

Examples 1. Any finite nonempty subset of $\mathbb{F}$ has max and min

2. For $\mathbb{F} = \mathbb{R}$ and $a < b$, denote

$[a,b] :=$ $(a,b) :=$

$[a,b) :=$ $(a,b] :=$

(a) $\max [a,b] = \max (a,b] =$ $\min [a,b] = \min [a,b) =$

Maximum and minimum

(b) $\max [a, b)$, $\max (a,b)$, $\min (a, b]$, $\min (a,b)$ do not exist

3. Recall $\max [0, \sqrt{2}] = \max \{x \in \mathbb{R} : 0 \leq x \leq \sqrt{2}\} =$

But $\max \{q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2}\}$

Upper / lower bound

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

<u>Def</u> If , then M is called an

of S and S is called bounded above

If , then m is called a

of S and S is called bounded below

S is called , if it is bounded above and bounded below

<u>Examples</u> 1. Intervals $[a,b]$, $[a,b)$, $(a,b]$, (a,b) are bounded:

any $m \le a$ is a lower bound, any $M \ge b$ is an upper bound

for these sets.

2. If $s_0 = \max S$, then any $M \ge s_0$ is an upper bound for S.

3. Sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are not bounded above.

Supremum and infimum

Let $\mathbb{F}$ be an ordered field and let $S \subset \mathbb{F}$, $S \neq \emptyset$

Def If S is bounded above and S has a

then we call it the of S,

If S is bounded below and S has a

then we call it the of S,

Examples 1. If $\max S$ exists, then (similarly inf)

2. $\sup\,[a,b] = \sup\,[a,b) = \sup\,(a,b] = \sup\,(a,b) = b$ (similarly for inf)

Completeness axiom

3. (a) $\mathbb{F} = \mathbb{R}$ $\quad \max [0, \sqrt{2}] = \max \{ x \in \mathbb{R} : 0 \leq x \leq \sqrt{2} \} =$

$\sup [0, \sqrt{2}] = \sup \{ x \in \mathbb{R} : 0 \leq x \leq \sqrt{2} \} =$

(b) $\mathbb{F} = \mathbb{R}$ $\quad \max \{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \}$

$\sup \{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \} =$

(c) $\mathbb{F} = \mathbb{Q}$ $\quad \max \{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \}$

$\sup \{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \}$

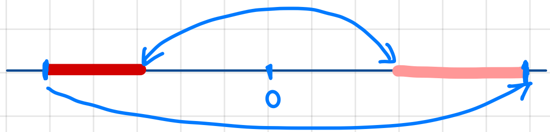
Completeness Axiom

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e., $\sup S$ exists and is a real number.

Satisfied by $\mathbb{R}$ (by definition), not satisfied by $\mathbb{Q}$.

Corollary 4.5

Let $S \subset \mathbb{R}$.



Proof

Denote $-S = \{-s : s \in S\}$.

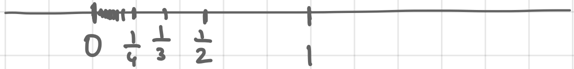
①: S bounded below $\Rightarrow$

②:

③:

Archimedean Property

- $\forall\ a>0\ \ \exists\, n\in\mathbb{N}$ s.t. $\frac{1}{n}<a$



- $\forall\ b>0\ \ \exists\, n\in\mathbb{N}$ s.t $n>b$

Thm 4.6 (Archimedean Property)

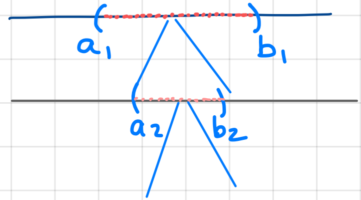
$\forall\ a>0, b>0\ \ \exists\, n\in\mathbb{N}$ s.t.

Proof: (by contradiction) Suppose AP is not true.

① $S:=\{an : n\in\mathbb{N}\}$

②

Denseness of $\mathbb{Q}$



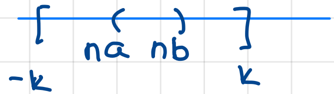
Thm 4.7 (Denseness of $\mathbb{Q}$)

$a<q<b$

$$(a, b \in \mathbb{R}) \wedge (a<b) \Rightarrow \exists\, q \in \mathbb{Q}\ (q \in (a,b))$$

Proof: Enough to show that $\exists\ m \in \mathbb{Z},\ n \in \mathbb{N}$ s.t.

$$a < \frac{m}{n} < b \iff an < m < bn$$



①

How to show that $\exists\, m \in \mathbb{Z}$ s.t. $an_0 < m < bn_0$?

Choose the smallest integer greater than an_0.

② $n_0 \max\{|a|,|b|\} > 0 \overset{AP}{\Rightarrow} \exists\ k$ s.t. $k \geq n_0 \max\{|a|,|b|\}$

$$\Rightarrow -k \leq n_0 a \leq n_0 b \leq k$$

③ $K := \{ j \in \mathbb{N} : -k \leq j \leq k,\ j > an_0 \}$, K finite and $K \neq \emptyset \Rightarrow \exists \min K =: m$

④ $m = \min K \Rightarrow m-1 \leq an_0 \Rightarrow m \leq an_0 + 1 \overset{①}{<} n_0 b \Rightarrow n_0 a < m < n_0 b$. ∎